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## PROCEEDINGS OF THE MEETING

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**Zoology.** — "*On the placentation of Sciurus vulgaris.*" By Dr. F. MULLER. (Communicated by Prof. A. A. W. HUBRECHT).

(Communicated in the meeting of September 29, 1906).

I. The very earliest stages. The ovule of Sciurus undergoes its first developmental stages in the oviduct. Meanwhile the bicornuate uterus has prepared itself for the reception of the ovule: underneath the single layer of epithelium the mucosa, which mesometrially remains very thin, has become very strongly thickened, so that an excentrical T-shaped slit is left open, the transverse part of which lies closest to the mesometrium. A special arrangement for the attachment of the ovules can nowhere be detected; a subepithelial zone is found to be richer in nuclei, however, than the loose connective tissue, separating this layer from the muscularis.

II. Pre-placentary stages (From the arrival of the ovule in the uterus until the first formation of the allantoid placenta).

The ovules fix themselves in varying numbers, to the right generally more than to the left, at about equal distances on the anti-mesometral (i.e. anti-placentary) uterine wall; they are fixed with their vegetative poles. A pellucid zone is absent, on the other hand the ovule becomes surrounded by a mass, formed from glandular secretions of cellular origin from the uterine wall.

The ovules grow pretty quickly, for the greater part by dilatation of the umbilical vesicle, which in these stages still forms the principal part of the ovule. It is remarkable that the area vasculosa remains so small, so that only entoderm and trophoblast form the wall of the germinal vesicle over the greater part of the umbilical vesicle.

The uterine wall shows intense activity during this stage. Many processes take place here in rapid succession and simultaneously. They all start from the spot where the ovule has settled, and from this point extend in all directions, successively reaching the spaces of the uterine horn, left open between the fixations of the ovules, as also the mesometrally situated parts; all these processes begin sub-epithelially, gradually penetrating deeper and deeper. These successive processes thus gradually give rise to dish-shaped layers of varying structure, surrounding the ovule at the anti-mesometral side and the character of which is most sharply pronounced in the points that are at the greatest distance from the mesometrium. By the extension of the anti-mesometral part of the long end of the

T-shaped slit, a broadening is brought about here, which, progressing more and more in the mesometral direction, finally produces a space, the cross-section of which presents a shape like that of a cone, truncated mesometrally by the old transverse part of the T, and bordered anti-mesometrally by a circular segment corresponding to the umbilical vesicle. The ovular chambers, formed in this way, have originated as the result of growth and extension of the anti-mesometral uterine wall, as a consequence of which the parts of the horn that connect them, are implanted at the mesometral side of the foetal chambers and at the same time are bent in this direction. The proliferation in the stroma tissue, beginning in the sub-epithelial layer, squeezes the mouths of the glands asunder. Later the epithelium in these latter degenerates, the walls agglutinate, finally only remnants of glands are found in the more normal stroma under the muscularis.

The processes by which the first formation of dish-shaped layers takes place (the existence of which is always of a relatively short duration, however) are the oedematous imbibition of the tissue and disintegration of cell-elements, accompanying the proliferation of the sub-epithelial multinuclear zone, the final result being a system of cavities, separated by thin cell-partitions and filled with the products of oedema and disintegration of tissue. This layer is externally surrounded by layers which form the transition to the still normal, although proliferating tissue, while at the opening they are more and more separated by products of a later process.

Very remarkable is the appearance at this time of "giant cells", plasma-lumps of different size, which assume a dark colour and contain many giant nuclei with a large nucleolus. Continuous layers or more isolated groups are found as individual differences, apparently. They lie mostly superficially, often connected with the trophoblast, not with the vessels. At first sight one would feel inclined to regard them as the foetal "suction roots" of the trophoblast, described for Spermophilus by REJSEK. Since all transitions are found between the mucosa tissue and these elements (in some cases even the transition having been followed up); since, moreover, they are found on the ovule, in course of being dissolved in the surrounding mass; since, on the other hand, in no case an origin from the trophoblast could be made probable, the giant cells are for this and other reasons in my opinion to be considered as a degenerative maternal formation, as a symplasm. They cannot be identified with the "monster cells" of MINOT, SCHOENFELD and others, described for the rabbit, since these elements are also found in Sciurus, only much later. Finally com-

parative anatomical considerations plead for my opinion (see below).

The uterine epithelium gradually disappears in all places where the germinal vesicle is in contact with it. There is never question of proliferation now.

Relatively soon already (even with a very extensive material the transitions are difficult to follow) a second stage sets in, in which by proliferation of the stroma cells, beginning from the surface, a dish is formed of cubical cells with granulated plasm (decidua), which extends more and more, while the above described cavities disappear, probably by resorption under influence of the pressure. In the mean time the decidua cells at the surface undergo further alterations and are resorbed (very likely by the umbilical vesicle, since in this and in the cells of the wall a similar substance can be found), so that a fine meshy texture is formed by the peripheral part of the cells remaining; by the pressure this meshy texture is compressed to a thin layer of lamellar structure, which in its youngest parts still shows the meshes. Vessels are not or scarcely found in the decidua. The separation between the decidua and the little or not changed subdecidual tissue outside it, is the limit to which the differentiating processes in the wall have progressed, at the same time approximately the limit, marking how far the agglutination of the germinal vesicle with the wall has advanced; it may therefore be called "differentiation limit".

At this time the above described giant cells become fewer and fewer in number, have an increasingly degenerate appearance and soon disappear altogether. At the mesometral side especially by proliferation of the epithelium an increase in number and size of crypts takes place (not of glands).

A second period in these pre-placentary stages is characterised for the ovule by the origin of the amnion etc. The growing embryo seeks place in an impression of the upper part of the umbilical vesicle, which becomes more and more accentuated. At the umbilical vesicle begins, starting again at the part that is most distant from the embryo, an outgrowth of the trophoblast cells with their nuclei, which process also continually advances towards the embryonic pole.

Characteristic for this stage are different processes beginning on the ovule as well as on the uterus in equatorial bands above the differentiation limit, and from here also proceeding mesometrally on all sides. For the ovule these processes consist in a proliferation of the trophoblast which will later cover the outer layer of the amnion fold; irregularly placed, dark, polygonal cells with distinct borders appear; on the surface of the trophoblast small cell-heaps rise every-

where. On the corresponding spot of the uterine wall a progressive process sets in; first: formation of crypts by proliferation of epithelium, at the same time thickening of the intermediate stroma; later by this process a ring has been formed, which everywhere projects a little into the lumen above the differentiation limit, dividing the cavity of the foetal chamber into a mesometrally situated placentary part, and an omphaloid part situated below it, while by this arrangement the meanwhile completed diplotrophoblast (chorion) with its very strongly thickened layer of trophoblast bulges out. The hyperplastic stroma of the projecting ring is everywhere well provided with crypts.

In the meantime this proliferation process has been closely followed by a regressive one; the epithelium begins to degenerate, first at the surface, later deeper and deeper in the formed crypts; plasm and nuclei become darker, more homogeneous, smaller; later the pycnotic nuclei dissolve in the plasm and a mass is formed, epithelial symplasm, in which finally greater and smaller vacuoles are evenly distributed. Everywhere short ramifications of epithelium have penetrated into the stroma, which soon degenerate. Also the stroma itself undergoes similar alterations later.

Wherever this degeneration has somewhat advanced, a third process sets in, likewise extending as a band: the thickened trophoblast penetrates with its ramifications some distance into the crypts, later also between these into the degenerated mass. Here and there the foetal mass thereby changes, after its elements have become enlarged and paler, into a syncytium, the nuclei of which contrast well with those of the maternal symplasm. In this connecting ring the syncytium soon disappears again; extension causes the trophoblast with its hollow ramifications, penetrating into the crypts, to become a single layer once more; exactly here the area vasculosa still remains for a time in connection with the trophoblast: everything pleads, in my opinion, for the hypothesis that this has to be considered as a rudiment, namely of an omphaloid placentation (Sorex, which among the Insectivora stands nearest the Rodents, shows a distinct omphaloid placentation).

The products of the crypts and glands, transsudates and symplasm masses, are shed over the cupola of the diplotrophoblast and probably are resorbed by this latter.

The vessels in the meantime proliferate strongly in the wall of the mesometral part of the foetal chamber between the crypts, which has remained unchanged yet; their wall consists as before of simple endothelium, without a perivascular sheath.

III. Placentary stages (After the beginning of the formation of the allantoid placenta). In the omphaloid part of the foetal cavity the wall is more and more attenuated by extension and resorption of tissue, although the layers may be recognised as before. The increase in size of the trophoblast cells of the umbilical vesicle, which had set in formerly, now leads to the formation of true "monster cells", the cellular body of which often shows concentric rings and other peculiarities, while the big nucleus often lies like a crescent round a vacuole. This process comes nearer and nearer the mesometrally situated formations. The entoderm, covering these monster cells, is very narrow and small-celled; where it covers the area vasculosa, it consists on the other hand of cubical, strong cells. In the umbilical vesicle a coagulated mass is always present. The large embryo more and more invaginates the upper part of the umbilical vesicle. Between the monstercells and the entoderm a sort of cuticle develops.

The processes, extending in equatorial bands, continally advance towards the mesometral pole of the foetal chamber, also in the partitions of the foetal chambers, so that they are more and more incorporated by these latter. In this manner extremely complicated pictures are formed, especially in cross-sections.

The dilatation now affects very strongly as well the placentary part of the foetal chambers as their mutual connecting pieces, so that the omphaloid part becomes smaller and smaller, while the formerly existing comb-shaped division between them disappears.

The progressive process finally reaches the mesometral pole of the placentary space and continually advances further into the connecting pieces of the foetal chambers: the still intact part of the wall, which at first had the shape of a cupola, later assumes the form of an 8, finally reduced to two round planes, which by the proliferation are more and more limited to the connecting pieces. The progressive process now forms crypts, which in other places are narrow and deep, but in the place of the placenta are broad and wide by dilatation and excessive proliferation of the stroma. The epithelium has many layers, its surface still rises everywhere in papillae. In the stroma not all the cells reach their full development as decidua cells simultaneously, so that a peculiar reticulated aspect is produced. Also the vessels increase.

In this soil now the degenerative process occurs, again advancing centripetally towards the mesometral pole. The epithelium becomes a symplasm, exactly like that described above, but this time more abundant and, everywhere covering the trophoblast. In the stroma

a conjunctival symplasm is formed, and, as was the case in the hyperplastic process, not everywhere simultaneously, so that small partitions of symplasm still surround more healthy groups. Outside the placentary trophoblast all this goes on until a single mass of epithelial and conjunctival symplasm is formed; within reach of the placentary trophoblast, however, the trophoblast has already penetrated before that time. During this degeneration also vessels are opened, so that extravasates are not rare now and altered blood is found against the trophoblast.

The third process by which the trophoblast is connected with the uterine wall, consists in the formation of hollow, one-layered invaginations in the crypts, which trophablast papillae are covered by caps of symplasm; between the crypts the trophoblast is extremely thin by extension, often irrecognisable, also when later the formation of giant cells had advanced thus far: these latter are then very long and narrow.

Finally the placentary trophoblast (which now forms if it were the keystone of the cupola of the diplotrophoblast and consists of a distinct basal layer of cells of one cell thickness and an often 20 cells thick layer above it) now lays itself everywhere against the papillae of the mesometral cupola of the placentary foetal chamber cavity, which papillae are in progress of being degenerated; the trophoblast papillae are likewise still covered by the symplasm, when between them this has already been resorbed.

Next comes the formation of a foetal syncytium from the superficial layers of the thickened placentary trophoblast, the process beginning above and centrally and proceeding centrifugally downwards; the nuclei which at first were dark and small, become larger and clearer, contain one big nucleolus and are clearly distinct from all maternal elements. This syncytium everywhere penetrates into the maternal tissue in strands, so that an intimate interweaving of maternal and foetal tissue results, proceeding centripetally into the papilla. Then everywhere "vacuoles" are formed in this mass (probably now for the first time at the expense of maternal tissue), which, when they become larger, bend the basal trophoblast layer (cytotrophoblast) inwards and finally fill with maternal blood. The allantois has meanwhile penetrated into the trophoblast papillae and is divided into small lobes by the growth of pairs of bulges of the cytotrophoblast. Foetal vessels soon penetrate freely into each lobule.

The primordium of the placenta as a whole has no round shape, the edges facing the connecting pieces of the foetal chambers are

concave to these sides, corresponding to the shape of the surface, here occupied by the progressive and regressive processes.

Gradually all the maternal tissue is replaced by foetal, so that finally the papillae which at first were entirely maternal, have become entirely foetal. Now the "vacuoles", surrounded by foetal syncytium dilate further (also grow at the expense of a foetal symplasm which now forms everywhere) and subdivide, a process, accompanied by constantly increasing separation of the allantoic villi by these cavities, containing maternal blood; the final result is that papilla-shaped lobes with secondary lateral lobules are formed, all separated by allantois-strands with foetal vessels; these are surrounded by the cytotrophoblast, which in its turn surrounds the "vacuoles" (now identical with cavities, containing maternal blood), enclosed by a layer of syncytium which at first is broad, later becomes gradually narrower. The placenta, originated in this way rests on a substructure of maternal tissue, composed of the same elements as formerly (decidua, etc.); the decidua-cells often grow out strongly, while the border between foetal and maternal tissue is in many places marked by a narrow streak of symplasm. The formation of trophoblastic giant cells gradually reaches also the supra-placentary parts, so that here also the enormous cells (later often free) lie in the maternal tissue.

Outside the placenta a stage soon is reached in which the progressive and regressive processes, described above, have attained their extreme limit. Superficially all has been changed into symplasm, only in the depth decidua-cells still exist, which meanwhile, since the degeneration does not reach to this depth, have become pretty large. The parts of the mucosa spared by these processes, are only the mucous membrane of the dilated connecting pieces, now entirely incorporated in the foetal chambers and whose mucosa, attenuated by extension, only possesses crypts still, that are squeezed flat, and a rather thick epithelium which for a part turns into symplasm. Against all these extra-placentary parts lies the extra-placentary trophoblast, now consisting entirely of giant cells which at present often get loose and then lie freely amid the decidua.

The embryo has, during its further growth, found place in the umbilical vesicle which gradually has become entirely invaginated and whose walls almost touch each other. The edges of the bowl, thus originated and containing the embryo, are not formed by the sinus terminalis; this latter lies further down in the inner wall. The small space in the umbilical vesicle is still filled with coagulating masses, while the entoderm, covering the area vasculosa, which now

often forms papillae, has still a very healthy appearance. The outer wall of this bowl never disappears. In its further growth the placenta reaches the edges of the bowl of the umbilical vesicle, later still it grows into it and coalesces with the inner wall: the sinus terminalis then lies halfway the thickness of the placenta, while a fold of the endoderm seems to have been incorporated into it.

In the last period of pregnancy, from the above described parts, left free by the progressive and regressive processes, epithelium grows between the degenerated and the normal part of the mucosa, perhaps joins with the meanwhile proliferating glandular remains in the depth: the umbilical vesicle is lifted off from the mucosa. Somewhat later this begins also all round the placenta, so that at the end of pregnancy this organ is more or less stalked and after parturition the greater part of the uterine wall is already provided with a new epithelium.

Comparative considerations. Among Rodents the investigation of the times at which various processes and organs of the ovule (not of the foetus) are found, leads to the following series: Sciurus — Lepus — Arvicola — Meriones — Mus — Cavia, in which the first has retained the most primitive forms, Lepus in many respects forms a transition to the last, in which more and more by new processes coming to the fore, the old, primitive ones are supplanted, mixed up and altered, in a word become nearly irrecognisable. Of this latter fact the study of the literature on the relation of ovule and uterus in Rodents, gives sufficient evidence; it also appears here how great a support is afforded by a comparative anatomical investigation; even, that various problems cannot be solved without its assistance.

The progression appears clearly in the pecularities of the umbilical vesicle in the various animals: in all the upper part is invaginated into the lower, with Sciurus not until late, with Cavia the process is among the first; the distal wall always remains with Sciurus, with Lepus it disappears late, with Cavia already quite at the beginning; the endoderm covers the inner wall already very early in Sciurus, very late in Mus, never entirely in Cavia.

In the same order the antimesometral fixation and the allantoid placenta occur earlier and together with these the trophoblast thickening, which causes them. It is exactly the remarkable pre-placentary processes which have been so carefully studied with Mus and Cavia, which by this replacing present the greatest difficulties.

With all Rodents the vegetative ovular pole becomes connected

with the anti-mesometral wall of the uterus. This connection only ends in Sciurus towards parturition, in Mus and Cavia already very early, in Lepus at an intermediate stage, by epithelium being pushed underneath from the connecting pieces of the foetal chambers.

In this fixation the umbilical vesicle is surrounded by proliferating mucosa tissue which later degenerates and is dissolved and resorbed by the ovule. The epithelium soon disappears after slight progressive changes, the stroma changes into decidua by very strong proliferation which in Mus, Cavia, etc. rises as reflexa round the ovule, corresponding with the smallness of the umbilical vesicle and consequently of the ovule. In accordance with an existing inclination, in the order of the above mentioned series, to replace nutrition by stroma products by maternal blood, the vascularisation of the decidua is very small in the squirrel, very strong in Cavia and correspondingly the extravasates, surrounding the ovule are very rare in Sciurus, common and abundant in Mus and Cavia.

In these processes in Sciurus maternal giant cells appear (symplasm) and later foetal ones, when the former have disappeared. In Lepus SCHOENFELD and others found the foetal giant cells (monster cells) already in earlier stages, in accordance with our series; all the cells then occurring are by him considered as foetal; probably, however, the maternal cells occur at the same stage and part of the described cells are of maternal, symplasmatic origin. With Mus both were found and distinguished by JENKINSON at much earlier stages, KOLSTER did not see the foetal ones, DUVAL not the maternal ones. So they must occur still earlier in Cavia; the foetal ones are then probably the proliferating "Gegenpolcellen" of v. SPEE, which perforate the zone at the vegetative pole; the maternal ones correspond to the products of the processes in the "Implantationshof" of v. SPEE. Also the disappearance of these formations takes place at an increasing rate (By all this it becomes clearer still that the comparison of Cavia and man by v. SPEE, which already from a phylogenetic point of view is hazardous, must be received with caution).

In the light of the comparative investigation these foetal "monster cells" may be considered as rudiments of an organ which was strongly developed in the ancestors of the Rodents.

In Sciurus the mass surrounding the ovule ("coagulum") consists especially of tissue products; these become less prominent in the order of the series and are replaced by blood.

Of the omphaloid placentation, which in Sciurus is already rudimentary, not much can be expected in the other members, although the study (until now neglected) of the morpho logy of the extra-

placentary parts of the foetal chamber might perhaps shed light on this subject.

The now following appearance of the allantoid placenta is found latest in Sciurus, earliest in Cavia. The tendency, increasing in the well-known order, to bring about as much as possible a nutrition without tissue products of the mucosa of the uterus and an allantoidean-placentary exchange between foetal and maternal blood, causes the processes, playing a part in placentation, to change: in Sciurus we still have a very strong hyperplasia of stroma-epithelium, later degeneration, disintegration and resorption with penetration of the trophoblast into this mass, all temporarily clearly distinct and relatively slow, in Cavia we find almost exclusively vascular proliferation, while proliferation and degeneration go hand in hand and the invasion of the trophoblast follows closely on these, this latter process not proceeding far and being soon finished (since the object: amener une hémorrhagie maternelle a être circonscrite par des tissus foetaux (DUVAL), is sooner reached). In the other animals all intermediate stages are found.

The later processes in the development of the placenta are in all different, although they are alike in principle: subdivision of cavities respectively vessels, containing maternal and foetal blood. The allantois remains passive, the foetal mass grows further and further round the allantois-ramifications, as it penetrates further into the cavity of the foetal chamber.

The formation of foetal giant cells proceeds with all Rodents over the whole trophoblast from the vegetative to the placentary pole; also the decidual cells become larger, so that also the giant cells, which in all have been found supra-placentary (as JENKINSON already stated for the Mouse), are partly of maternal, partly of foetal origin; with Sciurus the two always remain easy to distinguish.

The more or less isolated place, which according to the statements of authors, Lepus would in some respects occupy, will perhaps disappear, when the until now somewhat neglected study of the pre-placentary period will have been more extensively carried out (also in regard to the morphology of the foetal chamber).

Finally I have not become convinced that also for the morphology of the foetal chamber cavities the unity in the structural plan goes for all Rodents as far as is claimed by FLEISCHMANN; the difference in the statements I met with, will however perhaps disappear when all this has been studied with the aid of a more extensive material, although FLEISCHMANN's conceptions, for similar reasons, are certainly incorrect in their present shape.

**Botany.** — *"On the influence of the nectaries and other sugar-containing tissues in the flower on the opening of the anthers."* By Dr. W. Burck. (Communicated by Prof. F. A. F. C. Went.)

(Communicated in the meeting of September 29, 1906).

The consideration that the opening of the anthers is preceded by a very considerable loss of water [1]) and that with very many plants, e.g. *Compositae, Papilionaceae, Lobeliaceae, Antirrhineae, Rhinanthaceae, Fumariaceae* and further with all plants, chasmogamous as well as cleistogamous, which fertilise in the bud, this opening takes place within a closed flower and consequently cannot be caused by transpiration to the air, gave rise to the question whether perhaps the nectaries or other sugar-containing tissues in the flower, which do not secrete nectar outwardly, have influence on the withdrawal of water from the anthers.

My surmise that also among the plants whose anthers only burst after the opening of the flower, some would be found in which this process is independent of the hygroscopic condition of the air, was found to be correct. If the flowers are placed under a glass bell-jar, the air in which is saturated with water-vapour, the anthers of many plants burst at about the same time as those of flowers which are put outside the moist space in the open air.

This led me to arranging some experiments, yielding the following results:

1. If in a flower of *Diervilla* (*Weigelia*) *rosea* or *floribunda*, which is in progress of unfolding itself, one of the stamens is squeezed by means of a pair of pincers, so that the drainage of water from the stamen downwards is disturbed, the four anthers whose stamens have remained intact, spring open, but the fifth remains closed. With this plant it is not necessary to place the flower in a moist space; the same result is generally obtained if the flower remains attached to the plant.

If a flower is placed in the moist space together with the loose

[1]) This loss of water amounts e.g. with *Fritillaria imperialis* to 90 % of the weight of the anthers, with *Ornithogalum umbellatum* to 86 %, with *Diervilla floribunda* to 87 %, with *Aesculus Hippocastanum* to 88 %, with *Pyrus japonica* to 80 %, with different cultivated tulips 59—68 %, etc. With plants whose anthers burst in the flower, the loss is smaller; the anthers and the pollen remain moist then. With *Oenothera Lamarckiana* the loss amounts to 41 %, with *Canna hybrida grandiflora* to 56 %, with *Lathyrus latifolius* to 24 %.

anthers of another flower, those which are attached to the flower spring open; the loose ones don't. If only the corolla with the stamens attached to it is placed in the moist space, the anthers open as well as those of the complete flower. Consequently the nectary which is found in the middle of the flower at the side of the ovary, exerts no direct influence on the bursting of the anthers. If further a stamen is prepared in its full length and placed in the moist space together with some loose anthers, the anthers of the stamen burst, whereas the loose anthers remain closed.

From these experiments we infer that the anthers open under the influence of the stamen whether or not connected with the corolla. Now an investigation with FEHLING's solution shows that as well the stamen as the whole corolla and even the corollar slips, show the well-known reaction, indicating glucose.

Of *Digitalis purpurea* two of the anthers of a flower in the moist chamber, were separated from the corolla by an incision. The uncut anthers burst open, but the other two remained closed. A stamen prepared free over its full length causes the anther to burst in the moist chamber; loose anthers, on the other hand, remain closed.

An investigation with FEHLING's solution showed that here also the corolla contains glucose everywhere, but in especially large quantities where the stamens have coalesced with the corolla. Also the stamens are particularly rich in sugar over their entire length.

Of *Oenothera Lamarckiana*, the anthers of which burst already in the bud, a flower-bud was deprived of sepals and petals. One of the stamens was taken away from the flower in full length; of another stamen only the anther was removed. These three objects were placed together in the moist chamber. The anthers of the stamens which had remained connected with the tube of the calyx and those of the loose stamen sprang open; the loose anther, however, remained closed. An examination with FEHLING's solution gave the same result as was found above with *Digitalis*.

Similar experiments were made with the flowers of *Antirrhinum majus* L., *Lamium album* L., *Glechoma hederacea* L., *Salvia argentea* L., *Nicotiana affinis* Hort. and *sylvestris* Comes., and *Symphytum officinale* L., which all gave the same results, while with the flowers of *Ajuga reptans* L., *Stachys sylvatica* L., *Scrophularia nodosa* L., *Cynoglossum officinale* L., *Anchusa officinalis* L., *Echium vulgare* L., *Calceolaria pinnata*, *Hibiscus esculentus*, *Anoda lavateroides*, *Malva vulgaris* Tr., *Torenia asiatica*, *Corydalis lutea* Dc., *Colchicum autumnale* L., *Lysimachia vulgaris* L., *Atropa Belladona* L. and *Rhinanthus major* Ehrh. the experiments were restricted to showing that with

all of them the anthers spring open in a space, saturated with water-vapour. With all these plants the corolla and stamens react very strongly with FEHLING's solution.

*These experiments indicate that the water is withdrawn from the anthers by an osmotic action, having its origin in the glucose-containing tissue.*

I remark here that the presence of glucose — in so far as we may infer it from the precipitate of cuprous oxide after treatment with FEHLING's solution — in other parts of the flower than the nectaries proper and especially in the corolla, is a *very common* phenomenon (to which I hope to return later) and that it is not restricted to those flowers in which stamens and corolla have coalesced. There is rather question here of a quantitative difference than of a special property, peculiar to these flowers.

2. With *Stellaria media* the epipetalous stamens are mostly abortive, while of the episepalous ones only three have remained, as a rule. These three stamens bear at the base on the outside, a gland, secreting nectar.

If a flower is placed in the moist chamber and one of the stamens is injured with the pincers, the anthers of the uninjured stamens will afterwards burst, but the other remains closed. And when loose anthers from the flower are placed in the moist chamber, together with an intact flower, the loose anthers remain closed, while the anthers of the flower open. As well the petals as the stamens precipitate cuprous oxide from FEHLING's solution; also the tissue at the base of the sepals reacts with it. But the bursting of the anthers stands in no relation to this; if the petals are removed, this has no influence on the result of the just mentioned experiment.

The experiment indicates that the water is withdrawn from the anthers by the osmotic action, proceeding from the nectary.

In this connection it deserves notice that the nectaries of the epipetalous whorl and also those of the missing stamens of the episepalous whorl are abortive together with the stamens. The same is observed with *Cerastium semidecandrum* L., *C. erectum* L. and *Holosteum umbellatum* L.; here also the nectaries of the missing stamens have disappeared as a rule.

With the *Papilionaceae*, of which I investigated *Lupinus luteus L.*, *Lupinus grandifolius L.*, *Lathyrus odoratus L.*, *Lathyrus latifolius L.* and *Vicia Faba L.*, the anthers are known to open already in the closed flower. The petals precipitate cuprous oxide from FEHLING's solution, but exert no influence on the opening of the anthers. Flower buds of *Lathyrus latifolius* and *Lathyrus odoratus* were deprived of

their petals and placed in the moist chamber together with loose anthers. The loose anthers remained closed, but the others burst open.

In the same way as the flowers of *Stellaria media* and the mentioned *Papilionaceae*, behave with respect to the opening of the anthers in a space, saturated with water-vapour:

*Stellaria Holostea L., St. graminea L., Cerastium Biebersteinii C. arvense L., Cochlearia danica L., Sisymbrium Alliaria Scop., Crambe hispanica L., Bunias orientalis L., Capsella Bursa pastoris Mnch., Hesperis violacea L., H. matronalis L., Thlaspi arvense L., Alyssum maritimum Lam.*, and further *Lychnis diurna Sibth., Silene inflata* Sm. *Galium Mollugo L., Asperula ciliata Rochl., Campanula media L., C. latifolia L.*

*With all these plants the bursting of the anthers must, in my opinion, be ascribed to the influence of the nectaries.*

With *Hesperis* two large nectaries are found at the inner side of the base of the two short stamens and between these and the four long stamens. If a flower of *Hesperis violacea* or *H. matronalis L.*, after being deprived of its petals and sepals, is placed in the moist chamber, nearly always the four long stamens only burst; the other two remain closed.

It has been repeatedly observed that the secretion of nectar begins as soon as the stamens open.

In connection with what was stated above, one would be inclined to infer from this that flow of water from the anther causes the secretion of nectar. If, however, with *Stellaria media*, the anthers are removed before they have discharged water to the nectaries, one finds all the same the nectaries amply provided with honey, when the flower opens. The same may be observed in the male flowers of *Aesculus Hippocastanum*. In the still nearly closed flowerbud the nectary is dry yet. When the flower continues to open small drops of liquid are seen to appear on the surface of the nectary, still before the anthers extend halfway from the bud. These droplets increase in size as the anthers approach the moment in which they open. By weighing it may be proved that the anthers have already lost part of their original weight when the first droplets of nectar appear on the surface of the nectary. From this circumstance also one would be inclined to infer that the water of the anthers comes out again as nectar. When, however, from very young buds, whose nectary is not moist yet, the anthers are removed, yet at a later stage of development of the bud, secretion of nectar is found in them as in buds that have kept their anthers.

With *Fritillaria imperialis* I found the same; but here the secretion of nectar was not so abundant as in buds, the anthers of which had not been removed. In my opinion these observations indicate that the sugar, stored up in the nectaries or other sugar-containing tissues of the flower, at the moment when it begins to exert its osmotic action, attracts water not only from the anthers but also from other parts of its surroundings.

3. With the following plants the anthers remain closed in a space, saturated with water-vapour. In so far as they possess nectaries, these latter appeared to exert no influence on the bursting of the anthers.

*Ranunculus acris L., R. bulbosus L., Aquilegia vulgaris L., Clematis Vitalba L., Chelidonium majus L., Brassica oleracea L., Geranium molle L., G. Robertianum L., G. macrorhizum L., Geum urbanum L., Rubus caesius L., Philadelphus coronarius L., Heracleum Sphondylium L., H. lanatum Michx, Aegopodium Podagraria Spr., Carum Carvi L., Pimpinella magna L., Valeriana officinalis L., Ligustrum vulgare L. Majanthemum bifolium Dc.*, and *Iris Pseudacorus L.*

It is remarkable that *Brassica oleracea L.* forms an exception to what is otherwise generally observed with the Cruciferae; the position of the stamens with respect to the nectaries which secrete honey abundantly, would make us expect that in a moist chamber they would behave like the others. The same remark holds for the species of *Geranium*.

The secretion of nectar in the flower attracted the attention of various investigators many years before SPRENGEL published his view of the matter. Also after SPRENGEL, in the first half of the preceding century, it has many times been the object of investigation. All these investigators agreed in being convinced that, apart from the significance of the honey-secretion for the fertilisation of the flowers by the intervention of insects, to which SPRENGEL had drawn attention, the sugar-containing tissues and the secreted liquid were still in another respect useful to the plant.

After DARWIN had in 1859 brought to the front again SPRENGEL's observations on the biological significance of the various properties of the flower — which observations were falling more and more into oblivion — and had accepted their consequences by bringing them into relation on one hand with his conceptions about the necessity of cross-fertilisation for the maintenance of the vital energy

of the species, on the other hand with the theory of natural selection, the investigation of still another significance of the nectaries for the plant was for a long period entirely abandoned.

Not until 1878 this subject was again broached by BONNIER [1]) who, in his extensive paper on the nectaries, in which as well the anatomical as the physiological side of the problem were submitted to a very extensive investigation, proved that sugar-containing tissues in the flower and especially in the immediate vicinity of the ovary are not only found with plants which regularly secrete nectar during the flowering, but also with such plants as under normal conditions never secrete such a liquid. With these plants, which in the literature on flower biology are called "pollen flowers", since the insects find no nectar in them, he found as well sugar-containing tissues as in the so-called "insect flowers". Even with anemophilous plants he found "nectaires sans nectar", e. g. with *Avena sativa*, *Triticum sativum* and *Hordeum murinum*. A number of plants which under ordinary conditions of life contain no nectar, he could induce to nectar-secretion by placing them under conditions, favourable for this purpose.

At the end of his paper he reminds us that an accumnlation of reserve materials, wherever a temporary stagnation in the development exists, may be considered a very general and well characterised phenomenon. When a plant stops its further development at the end of its growing period, it has stored up reserve material in its subterranean, parts and when the seed has finished its development, it has accumulated nourishing substances in the endosperm or in the cotyledons of the embryo. These reserve materials, turned into assimilable compounds, then serve for the first nutrition of the newly formed parts.

He then arrives at the conclusion that in the vicinity of the ovary saccharose is stored up, and that this reserve substance after fertilisation and in the same proportion as the fruit develops, passes partly or entirely into the tissue of the fruit and into the seed, after having first been changed, under the influence of a soluble ferment, into assimilable compounds.

Investigation showed me also that the accumulation of saccharose as a reserve substance in the flower is a very common phenomenon [2]).

[1]) GASTON BONNIER. Les nectaires. Étude critique, anatomique et physiologique. Annales des sciences naturelles. Tome VIII. 1878.

[2]) On this point see also: PAUL KNUTH, Über den Nachweis von Nektarien auf chemischem Wege. Bot. Centralbl. LXXVI. Band, 1898, p. 76 and ROB. STÄGER, chemischer Nachweis von Nektarien bei Pollenblumen und Anemophilen. Beihefte zum Bot. Centralbl. Band XII. 1901, p. 34.

But besides the function, discovered by BONNIER and the significance of the secreted nectar for the fertilisation, it has become clear to me that as well the glucose, formed from saccharose, as the outwardly secreted nectar, are also in other respects of great importance to the plant. The observations, here communicated, point already to one very important function, i. e. *to enable the stamens to bring their pollen to the surface at the right time, independent of the hygroscopic condition of the air.*

I hope before long to be able to point out still another function.

The secretion of nectar now appears in another light. The view that it must be considered as an excretion of "a waste product of chemical changes in the sap" [1]), which in the course of time has become more marked through natural selection, as a useful adaptation for promoting cross-fertilisation, since this liquid was eagerly taken away by insects, has to give way to the conception that, preceding any adaptation, it has in its further development kept pace with the sexual organs.

**Anatomy**. — "*On the relation of the genital ducts to the genital gland in marsupials.*" By A. J. P. v. d. BROEK. (Communicated by Prof. L. BOLK).

(Communicated in the meeting of October 27, 1906).

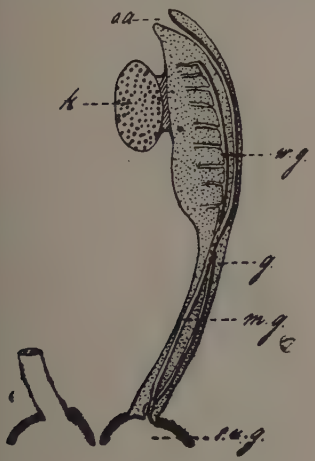
In the following communication the changes will be shortly described which the cranial extremities of the genital ducts in marsupials undergo during the development and their relations in regard to the genital gland. In more than one respect the ontogenetic development differs in these animals from what can be observed in other mammals.

It is especially a series of young marsupials of Dasyurus viverrinus in successive stadia of development from which the observations are derived. The preparations of other investigated forms (Didelphys, Sminthopsis crassicaudata, Phascologale pincillata, Trichosurus vulpecula, Macropus ruficollis) correspond however completely with the conditions we meet in Dasyurus.

In our description we start from a stadium schematically represented in figure 1 that still prevails for both sexes, (Dasyurus, Didelphys, Macropus). The genital gland (Figure 1 *k*) is situated at the medial

[1]) CH. DARWIN. Origin of species. Sixth Edition. 1872. Chap. IV, p. 73 and The effects of Cross and Selffertilisation. Edition 1876. Chap. X, p. 402.

side of the mesonephros and is attached to it by a narrow band (afterwards the mesorchium or mesovarium) (Fig. 1 *m*). The genital ducts are developed on their whole length. The Wolffian duct (*w.g.*) joins transversal mesonephridial tubules in the mesonephros but has no connection whatever as yet with the genital gland. The Müllerian duct (Figure 1 *m. g.*) commences with an ostium abdominale (*o. a.*) and runs as far as the region of the mesonephros is concerned at the lateral side of the Wolffian duct.




relation of the genital gland and genital ducts in an indifferent stadium.

*k.* genital gland.
*o.a.* Ostium abdominale tubae.
*g.s.* genital cord.
*w.g.* Wolffian duct.
*m.g.* Müllerian duct.
*s.u.g.* Sinus uro-genitalis.

Fig. 1.

We firstly will follow the transformations, which appear in the female sex. The first change is a reduction in the cranial part of the mesonephros. Here nothing is to be observed that points to a transformation of the mesonephridial tubules by renovation of the epithelium. The Wolffian duct meanwhile grows cranially, remains situated near the Müllerian duct, and moves then, passing archwise through the mesovarium, to the ovarium, penetrates in it and there ends blind (Figure 2 *w. g.*). The condition which issues from this I have demonstrated in Figure 2 (Dasyurus 40 m.m.).

Only now the reduction of the Wolffian duct begins. This occurs in such a way, that the medial part disappears; both at the cranial and at the caudal extremity, a remnant of the duct remains.

The cranial rudiment of the Wolffian duct is then found as a little tubule blind at both ends, which commences in the ovarium and can be traced till in the mesovarium. Figure 3 points out this little tubule as I have found it in several animals (Dasyurus, Sminthopsis) (Fig. 3 *w. g.*).

In how far the remnant of the Wolffian duct has relation to the

little tubules which I described and represented in the mesovarium of a fullgrown Petrogale penicillata, remains out of discussion here [1]).

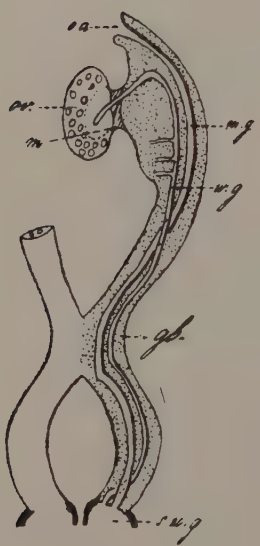


Fig. 2.

Relation of the genital ducts to the ovarium.

*ov.* Ovarium.
*m.* Mesovarium.
*o. a.* Ostium abdominale tubae.
*m. g.* Müllerian duct.
*w. g.* Wolffian duct.
*tr. c.* Transversal combination of both the genital cords.
*s. u. g.* Sinus uro-genitalis.

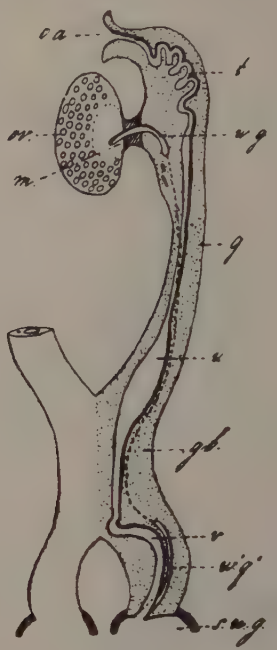


Fig. 3.

Relation of the genital ducts to the ovarium.

*ov.* Ovarium.
*m.* Mesovarium.
*o. a.* Ostium abdominale tubae.
*t.* Tuba Falloppie.
*u.* Uterus.
*v.* Vagina.
*w.g.* Remnant of the Wolffian ducts.
*w'.g'.* " " " " "
*g. s.* genital cord.
*tr. c.* Transversal combination of both the genital cords.
*s. u. g.* Sinus uro-genitalis.

In the male sex the Wolffian duct shows in the development of its cranial extremity, very much resemblance to that of the female sex. (Fig. 4 and 5).

During the reduction of the mesonephros the cranial extremity of

[1]) v. d. Broek, Untersuchungen über die weiblichen Geschlechtsorgane der Beuteltiere. Petrus Camper III.

the Wolffian duct grows forth and takes its course archwise through the mesorchium in the testicle. (Fig. 4 *w. g.*). Here is brought about in one place (Dasyurus) a connection with the future spermatic tubes, which are still present in the stadium of solid cords of cells.

The mesonephridial tubules disappear almost quite, so that at a certain stadium (Dasyurus viverrinus 53 m.m.) the Woffian duct, strongly grown forth in length, runs twisting through the mass of tissue, which must be considered as the epididymis, without any appearance of tubules in the form of the coni vasculosi.

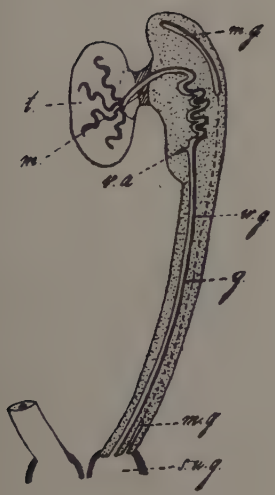



Fig. 4.

Relation of the genital ducts to the testicle.

*t.* Testicle.
*m.* Mesorchium.
*m.g.* Remnants of the Müllerian duct.
*w.g.* Wolffian duct (vas deferens).
*g.s.* Genital cord.
*s.u.g.* Sinus uro-genitalis.
*v.a.* Vas aberrans.

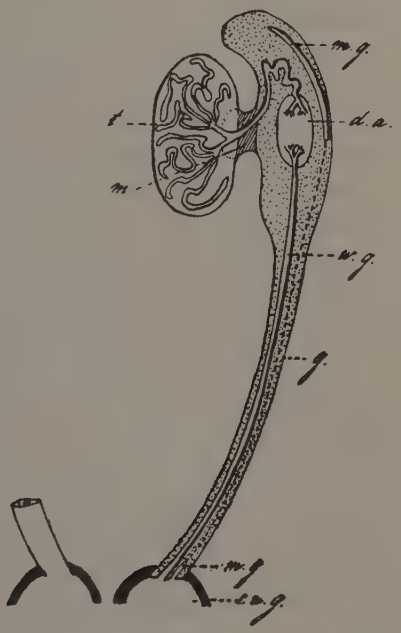



Fig. 5.

Relation of the genital duct to the testicle.

*t.* Testicle.
*m.* Mesorchium.
*m.g.* Remnants of the Müllerian duct.
*d.a.* Glandule part in the epididymis.
*w.g.* Wolffian duct (vas deferens).
*g.s.* Genital cord.
*s.u.g.* Sinus uro-genitalis.

Meanwhile the Müllerian duct is for the greater part reduced. The cranial extremity remains as a remnant of the duct either beginning with an ostium abdominale or not, and ending caudally blind in the epididymis tissue.

The change following on this consists therein that the spermatic tubes obtain a lumen and combine in one or two places (Didelphys) with the Wolffian duct grown into the testicle. In the epididymis a great many cell cords have meanwhile appeared in the course of the Wolffian duct (Vas epididymidis), out of which cell cords the little tubules of the epididymis will develop.

Of the Müllerian duct a rest has remained in the tissue of the epididymis, I have not observed rests of this duct in the form of hydatids. Neither did I find them mentioned in literature.

In the genital gland of the full grown animal I found that the connection of the testicle and epididymis is formed by a mesorchium, in which evidently a single tube forms the communication between the two parts (Didelphys, Halmaturus). Probably the same holds true for Hypsiprymnus, where, according to DISSELHORST [1]), the epididymis is a spindle shaped swelling in the course of the vas deferens.

About the microscopic structure of the testicle and epididymis I found in DISSELHORST the communication that it agrees with that of other animals. As on this immediately follows: "die Spermatogenese war in vollem Gange", it seems to me that this communication relates more to the structure of epithels of the tubules than to the nature of the connection of testicle and epididymis.

A comparison with what we find in other mammals shows us the following.

There now and then is to be observed in the female sex (at least in man) an excrescence of the cranial extremity of the Wolffian duct, which then becomes the tubo-parovarian tube, which was first described by ROTH [2]) and recognised by MIHALKOVICS [3]) as a part of the Wolffian duct. Where however in Marsupials the Wolffian duct penetrates into the genital gland, the tubo-parovarian tube of man remains in the Ligamentum latum.

For the male sex the following holds true.

A rete testis, whether it has to be considered as tubules, which have appeared afterwards, and must be considered as a second generation of tubuli seminiferi (COERT)[4]) or as homologa of the

[1]) R. DISSELHORST. Die männlichen Geschlechtsorgane der Monotremen und einiger Marsupialen.

SEMON's Zoölogische Forschungsreisen in Australiën und den Malayischen Archipel. 1904. p. 121.

[2]) Quoted by MIHALKOVICS.

[3]) MIHALKOVICS, Untersuchungen über die Entwickelungsgeschichte der Uro-genitalorgane der Cranioten.

Internat. Zeitschrift für Anatomie und Histologie. Bd. 2.

[4]) COERT, Over de ontwikkeling der geslachtsklier bij de z ogdieren. Diss. Leiden 1898.

"Markstränge" of the ovarium (MIHALKOVICS), or as tubules of the mesonephros grown into the tissue of the testicle (KOLLMANN) [1]) is not found in marsupials. If, during further development a network resembling the rete testis, arises in the marsupial testicle, it must be considered as a part which appears quite secondary.

The connection of the testicle and epididymis is not caused by a number of tubules of the mesonephros, transformed to vasa efferentia, but by a single tube which must be considered as a part of the Wolffian duct. For the conception that the connecting tube really is the Wolffian duct, the phenomena of development in the female sex can be cited together with those in the male animals. In the marsupials all the tubules of the mesonephros are reduced to minimal rests (vasa aberrantia). In the mass of tissue, which represents the so-called epididymis of these animals, a great number of tubes arise secondary, which afterwards probably possess as epididymis tubules the same function as the coni vasculosi in the epididymis of other mammals.

To explain the differences in the connection of the testicle and epididymis in marsupials and in other mammals, the following considerations seem to me to be of importance.

About the changes, which the mesonephros undergoes, by its connection with the testis, which connection furnishes the later vasa efferentia testis, we read the following in the extensive investigations of COERT [2]): In the proximal part of the Wolffian body where the Malpighian bodies are connected with the blastem of the rete testis, we see the glomeruli and the inner epithelium of the capsules disappearing gradually; after which the outer walls of these capsules form the blind extremities the mesonephridial tubules. The epithelium of the mesonephridial tubules also begins to have another aspect. Two kinds of processes occur here together: a number of epithelium cells are pushed out into the lumen and are destroyed, while on the other hand many new cells are formed (mitosis). With this the cells get another appearance both as concerns the nucleus and the protoplasm. The result is that at last the tubules of the menonephros are surrounded all over their extent, which formerly was not the case, by an uniform epithelium, formed by cylindrical cells, the nuclei ranged regularly at the basis of the cells. Whether the connection of these tubes with the Wolffian duct always remains unchanged during those transformations or is perhaps broken off and afterwards re-established in another place I have not been able to investigate.

[1]) KOLLMANN, Lehrbuch der Entwickelungsgeschichte des Menschen.

[2]) l.c. p. 96.

My opinion is that these investigations show that the vasa efferentia testis must not be considered as simple tubules of the mesonephros, but newly formed tubules, which use quite or for the greater part the way given to them by the tubules of the mesonephros. And that they are able to use this way finds its cause in this, that, according to FELIX and BÜHLER [1]) there is most probably no idea of a functioning of the mesonephros in monodelphic mammals, even not in the pig, where it is so strongly developed.

Not so in the didelphic mammals. Here the mesonephros does not only function embryonally, as is known, but still during the first period of the individual life. A separation of the mesonephros in two parts as is found in reptilia does not come about here.

The connection of the genital gland, especially of the testicle and its duct, the Wolffiian duct, could not, it may be supposed, in the stadium in which this connection will come about in other animals, be established in marsupials with the help of tubules of the mesonephros, because these had still to fulfill their excretory function.

Instead of this the connection could be established in such a way that the Wolffian duct grews out cranially and brings about itself the connection between the gland and its excretory duct.

At last the tubes, which occur secondary and independently ot the tubules of the mesonephros in the tissue of the epididymis, might be explained in the same way, i.e. as tubules which have the same signification as the coni vasculosi, but for the same reason do not originate on the bottom of tubules of the mesonephros but are separated from them both locally and temporarily.

Another view may be, that the tube which encroached in the genital gland, might not be the Wolffian duct but the most cranial tubule of the mesonephros so that in other words the so-called sexual part of the mesonephros in marsupials should be reduced. I do not believe that this conception is true, firstly because no separation between the tubules can be observed, and secondly because at the reduction of the mesonephros, as is mentioned above, in marsupials, nothing can be observed, as far as my preparations are concerned, of differences between the tubules of the mesonephros, what must surely be the case at a transformation of a tubule of the mesonephros to a connecting duct.

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[1]) FÉLIX und BÜHLER, Die Entwickelung der Harn und Geschlechtsorgane in HERTWIG's Handbuch der vergleichenden und experimentellen Entwickelungsgeschichte der Wirbeltiere.

**Physics.** — *"Supplement to Communication N°. 95a from the Physical Laboratory of Leiden, on the comparison of the thermo-element constantin-steel with the hydrogen thermometer".* By Prof. H. KAMERLINGH ONNES and C. A. CROMMELIN.

§ 14. *Corrected representation of the observations by a five term formula.*

As appears from note 1 the calculations in § 12 were made with not perfectly accurate values of the temperature at — 182° and in the same way the mean errors were derived from the assumption of those less accurate values. [1])

If the correct values of those temperatures for the calculations of the deviations $W-R_1$, $W-R_2$, $W-R_3$, $W-R_4$ in Table VIII, are used, the mean errors in microvolts become:

for formula (*B*I) ± 3.0
(*B*II) ± 3.4
(*B*III) ± 2.8 (2.5 without — 217°)
(*B*IV) ± 2.1

instead of

(*B*I) ± 2.8
(*B*II) ± 3.2
(*B*III) ± 2.6 (2.1 without — 217°)
(*B*IV) ± 1.8

which would also have been obtained if the observations at — 182° were excluded.

Now it was necessary to examine whether a repetition of the adjustment would diminish these mean errors. It appeared convincingly that this was not possible to an appreciable degree for (*B*I), (*B*II), (*B*III). It appeared possible for (*B*IV) to distribute the errors more equally. However, this only reduced the sum of squares from 26,57 to 26,14.

Instead of the coefficients $a_4$, $b_4$, $c_4$, $e_4$ and $f_4$ (see § 12) we get then

$$a'_4 = +4.32513 \qquad e'_4 = +0.023276$$
$$b'_4 = +0.409153 \qquad f'_4 = -0.0025269$$
$$c'_4 = +0.0015563$$

The deviations are given in Table IX under $W-R'_4$.

---

[1]) The correction amounted to 0°,081 in temperature or to 1.7 microvolt. in electromotive force.

§ 15. *Representation of the observations by means of a four term formula.*

We have now quite carried out the calculation of a formula of the form

$$E = a\left(\frac{t}{100}\right) + b\left(\frac{t}{100}\right)^2 + c\left(\frac{t}{100}\right)^3 + e\left(\frac{t}{100}\right)^4 \quad . \quad . \quad . \quad (C)$$

announced in note 2 of § 11, by the method of E. F. v. D. SANDE BAKHUYZEN, which proved to facilitate matters greatly again.

Four solutions ($C$) were found, viz. ($C$I), ($C$II), ($C$III) representing the observations down to — 253°, whereas in ($C$IV) only agreement down to — 217° has been sought for.

The coefficients in millivolts are the following:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $a$ | + 4.30192 | + 4.30571 | + 4.30398 | + 4.33031 |
| $b$ | + 0.357902 | + 0.366351 | + 0.363681 | + 0.421271 |
| $c$ | — 0.0250934 | — 0.0192565 | — 0.020071 | + 0.018683 |
| $e$ | + 0.0257462 | + 0.0270158 | + 0.0270044 | + 0.035268 |

The residuals have been given in tenth parts of microvolts in Table IX under $W—R_{CI}$, $W—R_{CII}$, $W—R_{CIII}$, $W—R_{CIV}$.

Just as with the five term formula, the residual at — 182° appeared also now greater than the others.

In calculation 3 it was tried to distribute the errors more equally, but the sum of squares appeared now to have increased.

The mean errors are if we include the observations down to — 253° for ($C$I), ($C$II), ($C$III), and only those down to — 217° for ($C$IV), for

| | |
|---|---|
| ($C$I) | ± 3.0 |
| ($C$II) | ± 2.9 |
| ($C$III) | ± 3.0 |
| ($C$IV) | ± 2.3 |

If — 182° is excluded, they become:

| | |
|---|---|
| ($C$I) | ± 2.7 |
| ($C$II) | ± 2.6 |
| ($C$IV) | ± 1.8 |

The mean errors of ($C$I), ($C$II), ($C$III) must be compared with those of ($B$I) and ($B$III), those of ($C$IV) with those of ($B$IV).

This comparison teaches that the four term formula for the representation of the observations may be considered to be almcst equivalent to the five term formula, and that therefore (this remark is in harmony with note 2 of § 13) for the calibration to — 217° the lowest number of temperatures for which observations are required, amounts to *four*. That three are not sufficient was already proved in § 11. This appears also clearly, when the mean error is determined, which rises to ± 7.6 microvolts for the three term formula.

## TABLE IX.

## DEVIATIONS OF THE CALIBRATION-FORMULAE FOR THE THERMO-ELEMENT CONSTANTIN-STEEL.

| I | II | III | IV | V | VI | VII |
|---|---|---|---|---|---|---|
| N°. | $t$ | $W-R_4'$ | $W-R_{CI}$ | $W-R_{CII}$ | $W-R_{CIII}$ | $W-R_{CIV}$ |
| 22 | — 29°.82 | — 12 | + 20 | + 15 | + 18 | — 19 |
| 24 and 20 | — 58.75 | + 16 | + 30 | + 26 | + 29 | + 4 |
| 21 and 23 | — 88.15 | + 14 | + 1 | + 1 | + 1 | + 1 |
| 1 and 17 | — 103.70 | — 6 | — 29 | — 28 | — 30 | — 20 |
| 16 and 18 | — 139.86 | + 1 | — 26 | — 24 | — 31 | — 17 |
| 19 | — 158.83 | — 10 | — 10 | — 10 | — 18 | — 10 |
| 3, 11 and 5 | [— 182.73] | + 26 | + 46 | + 44 | + 35 | + 34 |
| 4, 28 and 6 | — 195.19 | + 2 | + 23 | + 21 | + 12 | + 11 |
| 12, 27 and 7 | — 204.70 | — 20 | — 9 | — 11 | — 19 | — 18 |
| 26, 14, 13 and 8 | — 212.85 | + 24 | + 21 | + 21 | + 13 | + 21 |
| 29, 15 and 25 | — 217.55 | — 15 | — 30 | — 29 | — 37 | — 23 |
| 30 | — 252.93 | + 280 | 0 | + 20 | + 20 | + 150 |
| 31 | — 259.24 | + 485 | + 115 | + 141 | + 143 | + 313 |

**Mathematics.** — *"On a special class of homogeneous linear differential equations of the second order"*. By Prof. W. KAPTEYN.

The differential equation of LEGENDRE

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0$$

is satisfied by a polynomium $P_n(x)$ of the $n^{\text{th}}$ degree and by a function $Q_n(x)$ which may be reduced to the form

$$Q_n(x) = \int_{-1}^{1} \frac{P_n(z)\,dz}{x-z}$$

This function however is not determined for real values of the variable in the interval $-1$ to $+1$, the difference on both sides of this line being $2i\pi P_n(x)$.

In analogy to this we have examined the question: to determine all homogeneous linear differential equations of the second order of the form

$$R(x)\frac{d^2y}{dx^2} + S(x)\frac{dy}{dx} + T(x)y = 0$$

where the coefficients are polynomia in $x$, which possess the property that $y_1(x)$ being a first particular integral, the second integral may be written

$$y_2(x) = \int_{\alpha}^{\beta} \frac{y_1(z)\,dr}{x-z}$$

where $\alpha$ and $\beta$ represent two real values, supposing moreover that this integral has a meaning everywhere except on the line of discontinuity.

Let

$$R(x) = \sum_0^{\lambda} r_p x^p \quad , \quad S(x) = \sum_0^{\lambda} s_p x^p \quad , \quad T(x) = \sum_0^{\lambda} t_p x^p$$

then we obtain firstly the conditions

$$R(x) = (x-\alpha)(x-\beta)\,r(x) = (x-\alpha)(x-\beta)\sum_0^{\lambda-2} \varrho_p x^p$$

$$S(x) = R'(x) + (x-\alpha)(x-\beta)\sum_0^{\lambda-2} h_p x^p.$$

If now we put

$$G_p'' = \int_{\alpha}^{\beta} z^p y_1''(z)\,dz \quad , \quad G_p' = \int_{\alpha}^{\beta} z^p y_1'(z)\,dz \quad , \quad G_p = \int_{\alpha}^{\beta} z^p y_1(z)\,dz$$

and

$$M = -G_0''$$
$$N = (\alpha+\beta)\,G_0'' - G_1'' - 2G_0'$$
$$m = -G_0'$$
$$n = (\alpha+\beta)\,G_0' - G_1' - G_0$$

the further necessary conditions may be deduced from the equation

$$I + J = 0$$

where $I$ and $J$ represent the following polynomia of degree $\lambda-1$

$$I = \sum_{p=0}^{\lambda-1} (\varrho_p N + \varrho_{p-1} M)\, x^p + \sum_{p=0}^{\lambda-1} [\{h_p + (p+1)\,\varrho_{p+1}\}\, n + \{h_{p-1} + p\varrho_p\}\, m]\, x^p$$

$$J = \sum_{p=0}^{\lambda-1} (r_{p+1} G_p'' + s_{p+1} G_p' + t_{p+1} G_p) + x \sum_{p=0}^{\lambda-2} (r_{p+2} G_p'' + s_{p+2} G_p' + t_{p+2} G_p) + \ldots\ldots\ldots\ldots + x^{\lambda-2} \sum_{p=0}^{1} (r_{p+\lambda-1} G_p'' + s_{p+\lambda-1} G_p' + t_{p+\lambda-1} G_p) + x^{\lambda-1} \sum_{p=0} (r_{p+\lambda} G_p'' + s_{p+\lambda} G_p' + t_{p+\lambda} G_p).$$

From this we may easily deduce that if $\lambda = 2$, the most general differential equation of the second order possessing the property in question is

$$(x-\alpha)(x-\beta)\frac{d^2y}{dx^2} + [t_1(x-\alpha)(x-\beta) + 2x - \alpha - \beta]\frac{dy}{dx} + (t_0+t_1x)\,y = 0$$

where $\alpha$, $\beta$, $t_0$ and $t_1$ are arbitrary constants.

When $\lambda = 3$ the most general equation may be written

$$(x-\alpha)(x-\beta)(\varrho_1x+\varrho_0)\frac{d^2y}{dx^2} + (t_2x^3+s_2x^2+s_1x+s_0)\frac{dy}{dx} + (t_2x^2+t_1x+t_0)\,y = 0$$

Here however the ten constants must satisfy the following three conditions

$$s_1 + (\alpha+\beta)\,s_2 + (\alpha^2+\alpha\beta+\beta^2)\,t_2 = 2\varrho_0 + (\alpha+\beta)\,\varrho_1$$
$$s_0 - \alpha\beta s_2 - \alpha\beta(\alpha+\beta)\,t_2 = -(\alpha+\beta)\,\varrho_0 - 2\alpha\beta\varrho_1$$
$$(t_1 - s_2 + 2\varrho_1)\,G_0 - t_2\,G_1 = 0.$$

**Mathematics.** — *"Some formulae concerning the integers less than n and prime to n."* By Prof. J. C. KLUYVER.

The number $\varphi(n)$ of the integers $v$ less than $n$ and prime to $n$ can be expressed by means of the divisors $d$.

We have

$$\varphi(n) = \sum_{d/n} \mu(d)\, d', \qquad (dd' = n)$$

if we denote by $\mu(q)$ the arithmetical function, which equals 0 if $q$ be divisible by a square, and otherwise equals $+1$ or $-1$, according to $q$ being a product of an even or of an odd number of prime numbers.

This equation is a particular case of a more general one, by means of which certain symmetrical functions of the integers $v$ are expressible as a function of the divisors $d$.

This general relation may be written as follows [1])

$$\sum_{v} f(v) = \sum_{d/n} \mu(d) \sum_{k=1}^{k=d'} f(k\,d).$$

For the proof we have to observe that, supposing $(m, n) \sim D$, the term $f(m)$ occurs at the righthand side as often as $d$ in a divisor of $D$. Hence the total coefficient of the term $f(m)$ becomes

$$\sum_{d/D} \mu(d),$$

that is zero if $D$ be greater than unity, and 1 when $m$ is equal to one of the integers $v$.

We will consider some simple cases of KRONECKER's equation.

First, let

$$f(y) = e^{xy}.$$

The equation becomes

$$\sum_{v} e^{xv} = \sum_{d/n} \mu(d) \sum_{k=1}^{k=d'} e^{xkd} = \sum_{d/n} \mu(d)\, e^{xd}\, \frac{e^{xn} - 1}{e^{xd} - 1},$$

or because of

$$\sum_{d/n} \mu(d) = 0,$$

$$\sum_{v} e^{xv} = \sum_{d/n} \mu(d)\, \frac{e^{xn} - 1}{e^{xd} - d}.$$

If we write

$$\sum_{v} \frac{x e^{xv}}{e^{xn} - 1} = \sum_{d/n} \mu(d)\, \frac{x}{e^{xd} - 1},$$

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[1]) KRONECKER, Vorlesungen über Zahlentheorie. I, p. 251.

we may introduce the BERNOULLIAN functions $f_k(\theta)$, defined by the equation

$$\frac{e^{\theta x}-1}{e^x-1}=\theta+\sum_{k=1}^{k=\infty}x^k f_k(\theta),$$

and hence show that

$$\sum_\nu\left\{\frac{1}{n}+\sum_{k=1}^{k=\infty}x^k n^{k-1}f'_k\left(\frac{\nu}{n}\right)\right\}=\sum_{d/n}\frac{\mu(d)}{d}\left\{1-\tfrac{1}{2}xd+\frac{B_1}{2!}x^2d^2-\frac{B_2}{4!}x^4d^4+\ldots\right\}.$$

By equating the corresponding terms on the two sides we get

$$\sum_\nu f'_{2m}\left(\frac{\nu}{n}\right)=(-1)^{m-1}\frac{B_m}{2m!}\sum_{d/n}\mu(d)d^{-2m+1}$$

as a first generalisation of the relation

$$\sum_\nu \nu^0=\sum_{d/n}\mu(d)d.$$

Observing that we have

$$\sum_{d/n}\mu(d)d^{-2m+1}=\frac{1}{n^{2m-1}}\sum_{d/n}\mu(d)d^{2m-1},$$

there follows for two integers $n$ and $n'$, both having the same set of prime factors,

$$\frac{\sum_\nu f'_{2m}\left(\frac{\nu}{n}\right)}{\sum_{\nu'} f'_{2m}\left(\frac{\nu'}{n'}\right)}=\left(\frac{n'}{n}\right)^{2m-1}.$$

In the same way an expression for the sum of the $k^{\text{th}}$ powers of the integers $\nu$ may be obtained. Expanding both sides of the equation

$$\sum_\nu e^{x\nu}=\sum_{d/n}\mu(d)\frac{e^{xn}-1}{e^{xd}-1}$$

we find

$$\frac{1}{k!}\sum_\nu \nu^k=\sum_{d/n}\mu(d)d^k f_k(d').$$

Other relations of the same kind, containing trigonometrical functions are deduced by changing $x$ into $2\pi ix$.

From

$$\sum_\nu e^{2\pi ix\nu}=\sum_{d/n}\mu(d)\frac{e^{2\pi ixn}-1}{e^{2\pi ixd}-1}$$

we find by separating the real and imaginary parts

$$\sum_\nu \cos 2\pi x\nu=\tfrac{1}{2}\sin 2\pi xn\sum_{d/n}\mu(d)\cot \pi xd,$$

$$\sum_\nu \sin 2\pi x\nu=\sin^2\pi xn\sum_{d/n}\mu(d)\cot \pi xd.$$

In particular the first of these equations gives a simple result if we put $x = \frac{1}{n} + \varepsilon$, where $\varepsilon$ is a vanishing quantity. As the factor $\sin 2\pi x n$ tends to zero with $\varepsilon$ the whole right-hand side is annulled but for the term in which $d = n$.

So it follows that

$$\sum_{\nu} \cos \frac{2\pi\nu}{n} = \mu(n),$$

and we have $\mu(n)$, originally depending upon the prime factors of $n$, expressed as a function of the integers prime to $n$.

Similarly we may put in the second equation $x = \frac{1}{2n}$ and write

$$\sum_{\nu} \frac{\sin \pi\nu}{n} = \sum_{d/n} \mu(d) \cot \frac{\pi d}{2n}.$$

Still another trigonometrical formula may be obtained by the substitution $x = \frac{q}{n} + \varepsilon$. Let $D$ be the greatest common divisor of the integers $n$ and $q$, so that

$$n = n_0 D \quad , \quad q = q_0 D;$$

then as $\varepsilon$ vanishes, we have to retain at the right-hand side only those terms in which $qd$ is divisible by $n$, or what is the same the terms for which the complementary divisor $d'$ divides $D$.

Hence, we find

$$\sum_{\nu} \cos \frac{2\pi q\nu}{n} = \sum_{d'/D} \mu\left(\frac{n}{d'}\right) d' = D \sum_{d/D} \mu(n_0 d) \frac{1}{d}. \qquad (dd' = D)$$

Instead of extending the summation over all divisors $d$ of $D$, it suffices to take into account only those divisors $\delta$ of $n$, that are prime to $n_0$. In this way we find

$$D \sum_{d/D} \mu(n_0 d) \frac{1}{d} = \mu(n_0) D \sum_{\delta} \mu(\delta) \frac{1}{\delta},$$

and as the second side is readily reduced to

$$\mu(n_0) \frac{\varphi(n)}{\varphi(n_0)} = \mu\left(\frac{n}{D}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{D}\right)},$$

we obtain for any integer $q$, for which we have $(n, q) \sim D$,

$$\sum_{\nu} \cos \frac{2\pi q\nu}{n} = \mu\left(\frac{n}{D}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{D}\right)}.$$

Concerning the result

$$\sum_{\nu}' \cos \frac{2\pi\nu}{n} = \mu(n)$$

a slight remark may be made. To each integer $\nu$ a second $\nu' = n - \nu$ is conjugated; hence denoting by $\varrho_n$ an irreducible fraction $< \frac{1}{2}$ with the denominator $n$, we may write

$$2\sum \cos 2\pi\varrho_n = \mu(n),$$

and also

$$2\sum_{n \leqq g} \cos 2\pi\varrho_n = \sum_{n \leqq g} \mu(n).$$

Now for large values of $g$ the fractions $\varrho_n$ will spread themselves not homogeneously, but still with some regularity more or less all over the interval $0 - \frac{1}{2}$ and there is some reason to expect, that in the main the positive and the negative terms of the sum $\sum_{n \leqq g} \cos 2\pi\varrho_n$ will annul each other, hence the equation

$$2\sum_{n \leqq g} \cos 2\pi\varrho_n = \sum_{n \leqq g} \mu(n)$$

is quite consistent with the supposition of VON STERNECK, that as $g$ takes larger and larger values the absolute value of $\sum_{n \leqq g} \mu(n)$ does not exceed $\sqrt{g}$.

Another set of formulae will be obtained by substituting in KRONECKER's equation

$$f(y) = \log\left(e^{\frac{2\pi ix}{n}} - e^{\frac{2\pi iy}{n}}\right).$$

Thus we get

$$\sum_{\nu} \log\left(e^{\frac{2\pi ix}{n}} - e^{\frac{2\pi i\nu}{n}}\right) = \sum_{d/n} \mu(d) \sum_{k=1}^{k=d'} \log\left(e^{\frac{2\pi ix}{n}} - e^{\frac{2\pi ikd}{n}}\right),$$

or

$$\sum_{\nu} \log\left(e^{\frac{2\pi ix}{n}} - e^{\frac{2\pi i\nu}{n}}\right) = \sum_{d/n} \mu(d) \log\left(e^{\frac{2\pi ixd'}{n}} - 1\right)$$

and after some reductions

$$\sum_{\nu}' \log 2 \sin \frac{\pi}{n}(\nu - x) = \sum_{d/n} \mu(d) \log 2 \sin \frac{\pi x}{d}.$$

By repeated differentiations with respect to $x$ we may derive from this equation further analogies to the formula

$$\varphi(n) = \sum_{d/n} \mu(d)\, d'.$$

So for instance we obtain bij differentiating two times

$$\sum_{\nu} \frac{1}{\sin^2 \frac{\pi\nu}{n}} = \frac{1}{3} \sum_{d/n} \mu(d)\, d'^2$$

and by repeating the process

$$-\sum_{\nu} \left[ \frac{d^{2m}}{dy^{2m}} \log \sin y \right]_{y=\frac{\pi\nu}{n}} = \frac{B_m\, 2^{2m}}{2m} \sum_{d/n} \mu(d)\, d'^{2m},$$

a result included in the still somewhat more general relation

$$n^s \sum_{\nu}^{k=\infty} \sum_{k=1} \frac{1}{(nk-\nu)^s} = \zeta(s) \sum_{d/n} \mu(d)\, d'^s,$$

which is self evident from.

Returning to the equation

$$\sum_{\nu} \log 2 \sin y \frac{\pi}{n} (\nu - x) = \sum_{d/n} \mu(d) \log 2 \sin \frac{\pi x}{d},$$

we obtain as $x$ tends to zero

$$\sum_{\nu} \log 2 \sin \frac{\pi\nu}{n} = -\sum_{d/n} \mu(d) \log d.$$

In order to evaluate the right-hand side, we observe that for $n = p_1^{\alpha_1}\, p_2^{\alpha_2} \ldots$ we have

$$-\sum_{d/n} \mu(d) \log d = -\left[ \frac{d}{dy} (1 - e^{y \log p_1})(1 - e^{y \log p_2}) \ldots \right]_{y=0}.$$

So it is seen that, putting

$$-\sum_{d/n} \mu(d) \log d = \gamma(n),$$

the function $\gamma(n)$ is equal to zero for all integers $n$ having distinct prime factors, and that it takes the value $\log p$, when $n$ is any power of the prime number $p$.

Hence we may write

$$\prod_{\nu} 2 \sin \frac{\pi\nu}{n} = e^{\gamma(n)},$$

a result in a different way deduced by KRONECKER [1]).

Again in the equation

$$\prod_{\nu} 2 \sin \frac{\pi}{n} (\nu - x) = \prod_{d/n} \left( 2 \sin \frac{\pi x}{d} \right)^{\mu(d)}$$

we will make $x$ tend to $-\frac{n}{2}$.

If $n$ be odd, all divisors $d$ and $d'$ are odd also and we have at once

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[1]) KRONECKER, Vorlesungen über Zahlentheorie. I, p. 296,

$$\prod_{\nu} 2 \cos \frac{\pi \nu}{n} = \prod_{d/n} (-1)^{\frac{d'-1}{2}\mu(d)} = (-1)^{\frac{1}{2}\varphi(n)}.$$

If $n = 2m$ and $m$ be odd, we shall have $\varphi(m) = \varphi(n)$. Half the numbers $\varkappa$ prime to $m$ and less than $m$ will be equal to some integer $\nu$, the other half will be of the form $\nu - m$.

Hence we have

$$\prod_{\nu} 2 \sin \frac{2\pi\nu}{n} = (-1)^{\frac{1}{2}\varphi(n)} \prod_{\varkappa} 2 \sin \frac{2\pi\varkappa}{n} = (-1)^{\frac{1}{2}\varphi(n)} \prod_{\varkappa} 2 \sin \frac{\pi\varkappa}{m},$$

and therefore

$$\prod_{\nu} 2 \cos \frac{\pi\nu}{n} = (-1)^{\frac{1}{2}\varphi(n)} \frac{\prod_{\varkappa} 2 \sin \frac{\pi\varkappa}{m}}{\prod_{\nu} 2 \sin \frac{\pi\nu}{n}} = (-1)^{\frac{1}{2}\varphi(n)} e^{\gamma\left(\frac{n}{2}\right) - \gamma(n)}$$

Lastly, if $n = 2m$, and $m$ be even, we shall have $\varphi(m) = \frac{1}{2}\varphi(n)$. Now each of the numbers $\varkappa$ prime to $m$ and less than $m$ at the same time will be equal to some integer $\nu$ and to one of the differences $\nu - m$. Reasoning as before we have in this case

$$\prod_{\nu} 2 \sin \frac{2\pi\nu}{n} = (-1)^{\frac{1}{2}\varphi(u)} \prod_{\varkappa} \left(2 \sin \frac{2\pi\varkappa}{n}\right)^2 = (-1)^{\frac{1}{2}\varphi(n)} \prod_{\varkappa} \left(2 \sin \frac{\pi\varkappa}{m}\right)^2.$$

and therefore

$$\prod_{\nu} 2 \cos \frac{\pi\nu}{n} = (-1)^{\frac{1}{2}\varphi(n)} \frac{\prod_{\varkappa} \left(2 \sin \frac{\pi\varkappa}{m}\right)}{\prod_{\nu} 2 \sin \frac{\pi\nu}{n}} = (-1)^{\frac{1}{2}\varphi(n)} e^{2\gamma\left(\frac{n}{2}\right) - \gamma(n)}.$$

From the foregoing we may conclude as follows. If we put

$$\prod_{\nu} 2 \cos \frac{\pi\nu}{n} = (-1)^{\frac{1}{2}\varphi(n)} e^{\lambda(n)},$$

the arithmetical function $\lambda(n)$ is different from zero only when $n$ is double the power of any prime number $p$, in which case we have $\lambda(n) = \log p$.

Again we introduce here the irreducible fractions $\varrho_n$ less than $\frac{1}{2}$ with the denominator $n$; then denoting by $M(q)$ the least common multiple of all the integers not surpassing $q$ we may write

$$2 \sum_{n \leqq g} \log 2 \sin \pi\varrho_n = \sum_{n \leqq g} \gamma(n) = \log M(g),$$

$$2 \sum_{n \leqq g} \log 2 \cos \pi\varrho_n = \sum_{n \leqq g} \lambda(n) = \log M\left(\frac{g}{2}\right).$$

If we consider the quotient $log\, M(g) : log\, g$ as an approximate (but always too small) value of the number $A(g)$ of prime numbers less than $g$, to KRONECKER's result

$$A(g) = \frac{2}{log\, g} \sum_{n \leqq q} log\, 2 \sin \pi \varrho_n$$

we may add

$$A\left(\frac{g}{2}\right) = \frac{2}{log\, \frac{g}{2}} \sum_{n \leqq g} log\, 2 \cos \pi \varrho_n .$$

**Astronomy.** — "*Researches on the orbit of the periodic comet Holmes and on the perturbations of its elliptic motion.* IV." By Dr. H. J. ZWIERS. (Communicated by Prof. H. G. VAN DE SANDE BAKHUYZEN).

At the meeting of the Academy on the 27 January of 1906, a communication was made of my preliminary researches on the perturbations of the comet Holmes, during the period of its invisibility from January 1900 till January 1906, and also of an ephemeris of its apparent places from the 1st of May till the 31st of December 1906. This time again this computation led to its rediscovery. Owing to its large distance from the earth and the resulting faintness of its light, there seemed to be only a small chance for its observation during the first months. This proved to be true, as not before the 30th of August of this year, the Leiden observatory received a telegram, that the comet was found by prof. MAX WOLF at the observatory Koenigstuhl near Heidelberg, on a photograph taken in the night of the 28th of August of a part of the heavens where according to the ephemeris it ought to be found. The roughly measured place

$$\alpha = 61^\circ\ 51' \qquad \delta = +\ 42^\circ\ 28'$$

for $13^h\, 52^m 1$ local time, appeared to be in sufficient agreement with the calculation.

Afterwards the place of the comet has been twice photographically determined: on the 25th of September and on the 10th of October, and each time prof. WOLF was so kind, to communicate immediately to me the places as they had been obtained, after carefully measuring the plates. Although WOLF declared in a note to the observed

position of the 25th of September[1]) that the brightness had increased sufficiently, to make the comet visible in a powerful telescope, till now I did not hear, that any visual observation of the comet has been made. The three Heidelberg plates are therefore the only material that can be used for testing the elements and ephemeris given before.

I communicate here the results as I had the pleasure to receive them from prof. Wolf.

1. "Den Kometen Holmes habe ich auf der Platte von 28 August rechtwinklig an die 4 Sterne

A.G. Bonn 3456, 3462, 3472, 3493

angeschlossen, und die Messungen nach der Turner'schen Methode reduziert. Ich finde für 1906.0:

$\alpha = 4^h\ 7^m\ 34^s84$ $\qquad$ $\delta = +\ 42°\ 30'\ 59''9$

für die Aufnahmezeit: 1906 Aug. 28, $13^h\ 52^m1$ Kgst. Das äusserst schwache zentrale Kernchen wurde dabei eingestellt. Die Messung und Rechnung bezieht sich auf die mittleren Orte der 4 Sterne für 1906; sonst ist gar nichts angebracht."

(Note of the 5th of September 1906).

2. "Ich habe Ihren Kometen nochmals am 25 aufgenommen und finde ihn entschieden etwas heller. Den Ort nach Turner mit 3 Sternen (A.G. Bonn 3710, 3760, 3778) fand ich

1906 Sept. 25: $12^h\ 46^m0$ M.Z. Kgst.

$\alpha_{1906.0} = 4^h\ 32^m\ 10^s02$ $\qquad$ $\delta_{1906.0} = +\ 47°\ 34'\ 54''6$

Ich habe auch den letzten Ort (viz. of Aug. 28) mit nur 3 Sternen nochmals gerechnet (weil ein Stern sehr ungünstig war) und fand für 1906 August 28: $13^h\ 52^m1$ Kgst.:

$\alpha_{1906.0} = 4^h\ 7^m\ 35^s00$ $\qquad$ $\delta_{1906.0} = +\ 42°\ 30'\ 58''3$

Ich bin nicht sicher, ob diese Bestimmung aus 3 Sternen besser ist als die erst mitgeteilte."

(Note of the 29th of September 1906).

3. "Herr Dr. Kopff hat gestern den Ort einer Aufnahme vom 10 Okt. 1906 des Kometen Holmes ausgemessen .....

1906 Okt. 10: $9^h\ 1^m0$ Kgst.

$\alpha_{1906.0} = 4^h\ 34^m\ 48^s94$ $\qquad$ $\delta_{1906.0} = +\ 49°\ 54'\ 59''2$

Sterne: A.G. Bonn 3759, 3768, 3777 ..... Der Komet war

[1]) *Astron. Nachr.*, N°. 4123, S. 302.

diesmal schon recht schwach, wahrnehmbar schwächer als im September. Die Messung ist deshalb auch wohl etwas unsicherer."

(Note of the 13th of October 1906).

Concerning the observation on the 28th of August I preferred the position obtained from 3 reference stars.

For the reduction to the apparent place, I used as before in the ephemeris the constants of the Nautical Almanac, where the short period terms are omitted. Assuming for the parallax of the sun 8″80, I find for the Heidelberg Observatory the following constants:

$$\lambda = -0^h34^m54^s8$$
$$tg\,\varphi' = 0.06404$$
$$A = 9.58267$$
$$D = 0.82425$$

which are used for the computation of the parallax of the comet. The following table gives an account of the reduced observations.

TABLE I.

| No. | Red. on app. pl. | | Parallax | | Apparent geoc. place. | |
|---|---|---|---|---|---|---|
| | $\Delta\alpha$ | $\Delta\delta$ | $\Delta\alpha$ | $\Delta\delta$ | $\alpha$ | $\delta$ |
| | s | ″ | s | ″ | h m s | ° ′ ″ |
| 1 | + 1.888 | −8.55 | − 0.191 | +1.24 | 4 7 36.697 | +42 30 50.99 |
| 2 | + 2.929 | −8.57 | − 0.217 | +0.92 | 4 32 12.732 | +47 34 46.95 |
| 3 | + 3.593 | −7.51 | − 0.298 | +2.35 | 4 34 52.235 | +49 54 54.04 |

I used for comparison with the ephemeris my original computations, which contained in $\alpha$ as well as in $\delta$ one decimal place more than the published values. The computed places and their comparison with the observed positions, are given in the following table.

TABLE II.

| Local time | Aberration-time. | Comp. apparent place | | Observ.—Comp. | |
|---|---|---|---|---|---|
| | | $\alpha$ | $\delta$ | $\alpha$ | $\delta$ |
| | d | h m s | ° ′ ″ | s | ″ |
| Aug. 28.553602 | 0.013211 | 4 7 29.753 | +42 30 24.28 | +6.94 | +26.7 |
| Sept. 25.507699 | .012005 | 4 32 4.255 | +47 34 29.94 | +8.48 | +17.0 |
| Oct. 10.351449 | .011462 | 4 34 43.017 | +49 54 43.02 | +9.22 | +11.0 |

Together with the ephemeris I communicated a table containing the variations of the right ascension and the declination by a variation of the perihelion passage of + 4 or — 4 days. In comparing the above given values $O—C$ with the numbers of that table, it is evident that by a small negative variation of the perihelion passage, the agreement between observation and computation may be nearly attained, at least in $\alpha$. The deviations in $\delta$ cannot be used so well for that purpose, as the variations of $\delta$, resulting from a variation of $T$, are always much smaller than those of $\alpha$, and this is especially the case in the period during which these observations are made. Yet we may conclude from the table for $\Delta T = -4$ days that the positive errors in $\delta$ will not entirely disappear by a variation of $T$.

By means of a rough interpolation I derived from the 3 differences $O—C$ in right ascension the following corrections for the time of perihelion passage:

| | | | |
|---|---|---|---|
| Observ. of | Aug. 28: | $\Delta T = -0.0900$ | day |
| „ „ | Sept. 25: | — 0.0916 | „ |
| „ „ | Oct. 10: | — 0.0896 | „ |

In the average $\Delta T = -0.0904$ day, which at the rate of a mean daily motion of 517″448 corresponds to an increase of the mean anomalies of 46″8.

As a first step to correct the adopted elements of the orbit, I therefore computed the 3 places, in the supposition of an increase of the mean anomalies: 1° by 40″, 2° by 50″. I interpolated the following sun's co-ordinates (with reference to the mean equinox of 1906.0) from the Naut. Almanac.

TABLE III.

| 1906 | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| Aug. 28.540391 | — 0.9134887 | + 0.3947635 | + 0.1712510 |
| Sept. 25.495694 | — 1.0018399 | — 0.0318699 | — 0.0138250 |
| Oct. 10.339987 | — 0.9565810 | — 0.2616405 | — 0.1135029 |

For the reduction to the apparent places I added to the mean $\alpha$ of the comet: $f + g \sin(G + \alpha) \operatorname{tg} \delta$, to its mean $\delta$: $g \cos(G + \alpha)$. The following table contains the computed apparent places in the two suppositions.

TABLE IV.

| N°. | $\Delta M = +40''$ | | $\Delta M = +50''$ | |
|---|---|---|---|---|
| | $\alpha$ | $\delta$ | $\alpha$ | $\delta$ |
| | h m s | ° ′ ″ | h m s | ° ′ ″ |
| 1 | 4 7 35.758 | + 42 30 34.72 | 4 7 37.266 | + 42 30 37.38 |
| 2 | 4 32 11.451 | + 47 34 31.46 | 4 32 13.248 | + 47 34 31.85 |
| 3 | 4 34 51.050 | + 49 51 42.20 | 4 34 53.060 | + 49 51 41.99 |

A sufficient control is obtained here by comparing the values for $\Delta M = 0''$ (ephemeris), $\Delta M = +40''$ and $\Delta M = +50''$.

In comparing with the observed apparent places we obtain the following differences $O - C$:

TABLE V.

| N°. | $\Delta M = +40''$ | | $\Delta M = +50''$ | |
|---|---|---|---|---|
| | $\Delta\alpha$ | $\Delta\delta$ | $\Delta\alpha$ | $\Delta\delta$ |
| | s | ″ | s | ″ |
| 1 | + 0.939 | + 16.27 | − 0.569 | + 13.61 |
| 2 | + 1.281 | + 15.49 | − 0.516 | + 15.10 |
| 3 | + 1.185 | + 11.84 | − 0.825 | + 12.05 |

By means of interpolation between the values of $\Delta\alpha$ we find as resulting value for $\Delta M + 46''412$, leaving the following errors:

| N°. | $\Delta\alpha$ | $\Delta\delta$ |
|---|---|---|
| | s | ″ |
| 1 | − 0.03 | + 14.7 |
| 2 | + 0.13 | + 15.2 |
| 3 | − 0.10 | + 11.9 |

From this follows that by a variation of $M$ alone, the differences $O - C$ in $\alpha$ can be reduced to very small quantities, but this is not the case with the differences in $\delta$. It could be seen beforehand

that no further improvement could be expected from alterations in $\pi$, $\varphi$ or $\mu$; at the end I will add a few words on these elements. So we must try to bring it about by variations in the position of the plane of the orbit, viz. of $i$ and $\Omega$, and for this reason I determined the relation between those elements and the computed places of the comet. As from the two suppositions $\Delta M = +50''$ seems to be nearer to the truth, I computed the apparent places of the comet: for $\Delta M = +50''$ $\Delta i = +10''$ and $\Delta\Omega = 0$ and also for $\Delta M = +50''$ $\Delta i = 0$ $\Delta\Omega = -10''$. Probably a somewhat larger value of $\Delta\Omega$ had been more convenient. The following table gives the variations of $\alpha$ and $\delta$ in the two cases.

TABLE VI.

| N°. | $\Delta i = +10''$ | | $\Delta\Omega = -10''$ | |
|---|---|---|---|---|
| | $\Delta\alpha$ | $\Delta\delta$ | $\Delta\alpha$ | $\Delta\delta$ |
| | s | " | s | " |
| 1 | − 0.149 | + 10.00 | + 0.040 | + 1.26 |
| 2 | − 0.108 | + 11.95 | + 0.067 | + 0.83 |
| 3 | − 0.111 | + 12.88 | + 0.080 | + 0.56 |

The numbers from the tables V and VI give the following values of the differential quotients of $\alpha$ and $\delta$ with respect to $M$, $i$ and $\Omega$, which will be used as coefficients in the equations of condition.

| | Aug. 28 | Sept. 25 | Oct. 10 |
|---|---|---|---|
| $\frac{\partial\alpha}{\partial M}$ | + 0.1508 | + 0.1797 | + 0.2010 |
| $\frac{\partial\delta}{\partial M}$ | + 0.266 | + 0.039 | − 0.021 |
| $\frac{\partial\alpha}{\partial i}$ | − 0.0149 | − 0.0108 | − 0.0111 |
| $\frac{\partial\delta}{\partial i}$ | + 1.000 | + 1.195 | + 1.288 |
| $\frac{\partial\alpha}{\partial\Omega}$ | − 0.0040 | − 0.0067 | − 0.0080 |
| $\frac{\partial\delta}{\partial\Omega}$ | − 0.126 | − 0.083 | − 0 056 |

For $\alpha$ the second of time and for the others quantities the second of arc have been adopted as unities. I multiplied the equations of condition for $\alpha$ by $15 \cos \delta$, and instead of $\Delta \Omega$ I introduced $\frac{\Delta \Omega}{10}$ as unknown quantity.

*Equations of condition.*

*a.* From the Right ascensions:

$$0.22202\ \Delta M + 9.21681_n\ \Delta i + 9.64568_n \frac{\Delta \Omega}{10} = 0.79873_n$$
$$0.25966\ \text{„} + 9.03853_n\ \text{„} + 9.83118_n\ \text{„} = 0.71776_n$$
$$0.28811\ \text{„} + 9.03023_n\ \text{„} + 9.88800_n\ \text{„} = 0.90136_n$$

*b.* From the Declinations:

$$9.42488\ \Delta M + 0.00000\ \Delta i + 0.10037_n \frac{\Delta \Omega}{10} = 1.13386$$
$$8.59106\ \text{„} + 0.07737\ \text{„} + 9.91908_n\ \text{„} = 1.17898$$
$$8.32222_n\ \text{„} + 0.10992\ \text{„} + 9.74819_n\ \text{„} = 1.08099$$

The coefficients are written logarithmically; the second members are taken from column 4 and 5 of table V, and therefore to $\Delta M$, found from these equations, the correction $+ 50''$ has still to be applied.

From the above equations of condition we derive in the ordinary way the following normal equations:

$$+ 9.9278\ \Delta M - 0.39596\ \Delta i - 3.8260 \frac{\Delta \Omega}{10} = - 31.495$$
$$- 0.39596\ \text{„} + 4.1375\ \text{„} - 2.7434\ \text{„} = + 49.637$$
$$- 3.8260\ \text{„} - 2.7434\ \text{„} + 3.8423\ \text{„} = - 23.951$$

These equations are much simpler if we introduce besides $\Delta M$, only one of the two unknown quantities. If we try e.g. to represent the observations only through variations of $M$ and $i$ we have not only $\Delta \Omega = 0$ but the third equation falls out entirely.

1. *Solution for* $\Delta \Omega = 0$.

The results are:

$$\Delta M = - \ 2''\,7042$$
$$\Delta i = + 11.74$$

and the remaining errors:

| | | |
|---|---|---|
| 1. | $\Delta \alpha = + 0^s014$ | $\Delta \delta = + 2''59$ |
| 2. | $= + 0.097$ | $+ 1.18$ |
| 3. | $= - 0.151$ | $- 3.13$ |

2. *Solution for* $\Delta i = 0$.

In this case we find:

$$\Delta M = - \quad 9''0461$$
$$\Delta \Omega = - \; 2'32''41$$

and for the remaining errors:

| | | |
|---|---|---|
| 1. | $\Delta\alpha = + 0^s185$ | $\Delta\delta = - 3''18$ |
| 2. | $+ 0.089$ | $+ 2.80$ |
| 3. | $- 0.226$ | $+ 3.32$ |

3. *Solution with* 3 *unknown quantities:*

The results are:

$$\Delta M = - \quad 5''3045$$
$$\Delta i = + \quad 7.32$$
$$\Delta \Omega = - \; 1' 2.90$$

and according to the equations of condition there remain the following differences Obs.—Comp.

| | | |
|---|---|---|
| 1. | $\Delta\alpha = + 0^s088$ | $\Delta\delta = - 0''23$ |
| 2. | $+ 0.095$ | $+ 1.34$ |
| 3 | $- 0.181$ | $- 1.01$ |

As we see the solution with $\Delta\Omega = 0$ and that with $\Delta i = 0$ satisfy the observations fairly well, the first one somewhat better, especially in right ascension. Still we cannot deny that in the values Obs.—Comp. of $\delta$ in both solutions, there exists a systematic variation. On account of that I prefer for the present the solution with 3 unknown quantities, where such a systematic variation doesnot appear. I therefore take the following elements as the most probable for the return in 1906:

Epoch 1906 January 16.0 M.T. Greenw.

$$M_0 = 1266456''838$$
$$= 351°47'36''838$$
$$\mu = 517''447665$$
$$\log a = 0.5574268$$
$$T = 1906 \text{ March } 14.09401$$
$$\varphi = 24°20'25''55$$
$$e = 0.4121574$$
$$i = 20°49'\ 0''62$$
$$\pi = 346\ \ 2\ 31.63 \quad \Big\} \ 1906.0$$
$$\Omega = 331\ 44\ 37.85$$

Yet it is evident that the accuracy of these elements is not equal to the accuracy of those I could derive for previous returns of the comet. In the first place the observations include only a period of 43 days, in which the heliocentric motion of the comet with its large perihelion distance was not even 12°. Secondly three observations with their inevitable errors are in general only sufficient to obtain a mere approximate idea of the orbit. We must admire the ability and accuracy of the Heidelberg astronomers, who, from measurements on a short focal photographic plate taken of a still wholly invisible nebula, could deduce the position of the comet with an accuracy that could be compared to that of micrometer measurements of objects several hundred times brighter. Still we must bear in mind that the rejection of only one of the 4 reference stars on the plate of the 28th of August, had an influence of $0^s16$ in $\alpha$ and $1''6$ in declination, or of $2''39$ in arc of a great circle.

As a test to my calculations, I derived the 3 places finally by direct computation from the obtained elements.

*Heliocentric aequatorial co-ordinates:*

$$x = [9.993\ 7648.63]\ \sin(v + 77°37'28''36)$$
$$y = [9.876\ 2140.59]\ \sin(v - 20\ 58\ 46.82)$$
$$z = [9.832\ 7020.56]\ \sin(v - 1\ 46\ 46.76)$$

The following table contains the computed apparent places of the comet and the differences Obs.—Comp.

TABLE VII.

| N0. | $\alpha$ | $\delta$ | $\Delta\alpha$ | $\Delta\delta$ |
|---|---|---|---|---|
| | h m s | ° ′ ″ | s | ″ |
| 1 | 4 7 36.602 | + 42 30 51.32 | + 0.095 | — 0.33 |
| 2 | 4 32 12.633 | + 47 34 45.69 | + 0.099 | + 1.26 |
| 3 | 4 34 52.412 | + 49 54 55.19 | — 0.177 | — 1.15 |

The agreement between these differences found directly, and the quantities obtained by substitution in the equations of condition forms a sufficient control on the whole computation.

*The elements $\mu$, $\pi$ and $\varphi$.*

The elements from which the ephemeris for 1906 has been derived are those given in "Système VII" p. 78 of my Deuxième Mémoire, reduced to 1906 by applying the perturbations, arising from the action of Jupiter. The mean error of the obtained value for $\mu$ is so

small, that although not absolutely impossible, it is hardly probable that the correction obtained for the mean anomaly should have been caused totally or for the greater part by an error in $\mu$. Taking the obtained $\Delta M$ for the 25th of Sept. we get:

$$\Delta \mu = + \frac{44'' 6955}{2662.50} = + 0'' 016787$$

and thus the real error of $\mu$ should be 67 times the mean one. Adopting this correction of $\mu$, the mean anomalies for the 28th of August and the 10th of October would be only $0'' 469$ smaller and $0'' 249$ greater than the adopted ones.

It is more probable that the correction of $M$ arises from neglected perturbations of that element by Saturn. This perturbation is given by the formula

$$\Delta M = \int_{t_0}^{t} \frac{dM}{dt}\, dt + \int\int_{t_0}^{t} \frac{d\mu}{dt}\, dt^2.$$

Even if instead of the sum of the values each term was known separately it would be equally impossible to conclude from the value of the double integral, the final value of $\int \frac{d\mu}{dt}\, dt$, or the correction of $\mu$ for 1906. Observations during a much longer period can only decide in this case.

Something like this holds for $\pi$ and $\varphi$. During the short period of the observations, we may even substitute for a part of the correction $\Delta M$ corresponding variations of $\pi$ and $\varphi$. If we keep to the plane of the orbit, the apparent place, except for small variations in the radius-vector (of little influence near the opposition), depends wholly on the longitude in the orbit, or on

$$l = \pi + v.$$

So we can apply small variations to the elements without varying perceptibly the computed positions, if only

$$\Delta l = \Delta \pi + \Delta v = 0$$

or

$$\Delta \pi = - \Delta v.$$

This relation provides us with the means to throw a part of the correction found for $M$ on $\pi$ or on $\varphi$ or on both together. In the first case we have to satisfy the equation

$$\Delta \pi = - \frac{\partial v}{\partial M} \Delta M.$$

We can derive the values of $\frac{\partial v}{\partial M}$ directly from the comparison of the two former computations with $\Delta M = + 40''$ and $\Delta M = + 50''$. And so I find for the three dates of the observations:

$$\Delta M = - 0.506\ \Delta\pi$$
$$- 0.549\ \Delta\pi$$
$$- 0.573\ \Delta\pi$$

If we keep $\pi$ constant and want to substitute a part of the correction of $M$ by a variation of $\varphi$, we must satisfy the relation

$$\Delta v = 0$$

or

$$\Delta M = \left(\frac{\partial M}{\partial \varphi}\right)_{v \text{ const.}} \Delta\varphi.$$

I derived the values of $\left(\frac{\partial M}{\partial \varphi}\right)_{v \text{ const.}}$ by computing from the three values of $v$, with a varied excentricity, the corresponding values of the mean anomaly. Hence I got for the three observations:

$$\Delta M = - 1.040\ \Delta\varphi$$
$$- 1.186\ \Delta\varphi$$
$$- 1.260\ \Delta\varphi$$

Although the coefficients as well those of $\Delta\pi$ as of $\Delta\varphi$ show a small variation in the influence of the corrections of the elements on the three positions, practically this influence differs too little from that of a constant variation of $M$ to allow a determination of $\Delta M$, $\Delta\varphi$ and $\Delta\pi$ separately from the three observations.

*Leiden*, November 1906.

**Mathematics.** "*On the locus of the pairs of common points and the envelope of the common chords of the curves of three pencils.*" (1st part). By Dr. F. SCHUH. (Communicated by Prof. P. H. SCHOUTE).

1. *Given three pencils* $(C_r)$, $(C_s)$, $(C_t)$ *of plane curves of degree* $r$, $s$, $t$. *To find the locus* $L$ *of the pairs of points through which passes a curve of each of those pencils.*

Let $P$ and $P'$ be the points of such a pair. When determining the locus we shall notice but those points $P$ and $P'$ which are for each couple of pencils *movable* points of intersection (i. e. points not necessarily coinciding with the basepoints), a distinction to be made only when the pencils have common basepoints. The locus $L$ arrived

at in this way we shall call the *locus proper*, to distinguish it from the *total locus* to be arrived at by allowing one of the points $P$ and $P'$ to be a fixed point of intersection of two of the pencils.

Suppose the pencils $(C_s)$ and $(C_t)$ show $\alpha$ fixed points of intersection and that this number amounts to $\beta$ for the pencils $(C_t)$ and $(C_r)$ and to $\gamma$ for the pencils $(C_r)$ and $(C_s)$.

The degree $n$ of $L$ is determined from its points of intersection with an arbitrary straight line $l$. On $l$ we take an arbitrary point $Q_{rs}$ and through $Q_{rs}$ we let a $C_r$ and a $C_s$ pass, which cut each other besides in the basepoints and in $Q_{rs}$ still in $rs - \gamma - 1$ points. Through each of these points we let a curve $C_t$ pass. These $rs - \gamma - 1$ curves $C_t$ cut $l$ in $t(rs - \gamma - 1)$ points $Q_t$, which we make to correspond to the point $Q_{rs}$. To find reversely how many points $Q_{rs}$ correspond to a given point $Q_t$ of $l$ we take on $l$ an arbitrary point $Q_r$ through which we allow a $C_r$ to pass cutting the $C_t$ through $Q_t$ in $rt - \beta$ points differing from the basepoints. Through each of those points we allow a $C_s$ to pass, of which the points of intersection with $l$ shall be called $Q_s$. To a point $Q_r$ now correspond $s(rt - \beta)$ points $Q_s$ and to a point $Q_s$ correspond $r(st - \alpha)$ points $Q_r$. The $2rst - \alpha r - \beta s$ coincidences $Q_r Q_s$ are the $t$ points of intersection of $l$ with the $C_t$ passing through $Q_t$ and the points $Q_{rs}$ corresponding to $Q_t$, whose number therefore amounts to $2rst - \alpha r - \beta s - t$.

So between the points $Q_{rs}$ and $Q_t$ of $l$ we have a $(rst - \gamma t - t, 2rst - \alpha r - \beta s - t)$-correspondence. The $3rst - \alpha r - \beta s - \gamma t - 2t$ coincidences are the points of intersection of $l$ with $L$ and the points of intersection of $l$ with the curve of contact of the pencils $(C_r)$ and $(C_s)$, i. e. the locus of the points of contact of the curves $C_r$ and $C_s$ touching each other. If there are two systems of curves $(\mu_1, \nu_1)$ and $(\mu_2, \nu_2)$ [1]), the order of that curve of contact is

$$\mu_1\nu_2 + \mu_2\nu_1 + \mu_1\mu_2 \text{ [2])}.$$

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[1]) A system of curves $(\mu, \nu)$ is a simply infinite system of curves, of which $\mu$ pass through an arbitrarily given point and $\nu$ touch an arbitrarily given straight line.

[2]) This order is found by counting the points of intersection with an arbitrary line $l$. To this end we consider the envelope of the tangents of the curves of the system $(\mu_1, \nu_1)$ in its points of intersection with $l$; this envelope is of class $\mu_1 + \nu_1$, the tangents of that envelope passing through an arbitrary point $Q$ of $l$ being the tangents in $Q$ to the $\mu_1$ curves of the system through $Q$ and the line $l$ counting $\nu_1$ times. In like manner does the system $(\mu_2, \nu_2)$ give an envelope of class $\mu_2 + \nu_2$. The $(\mu_1 + \nu_1)(\mu_2 + \nu_2)$ common tangents of both envelopes are the line $l$ counting $\nu_1\nu_2$ times and $\mu_1\mu_2 + \mu_1\nu_2 + \mu_2\nu_1$ other lines whose points of intersection with $l$ indicate the points of intersection of $l$ with the curve of contact. For a deduction with the aid of the symbolism of conditions see SCHUBERT, "Kalkül der abzählenden Geometrie", p. 51—52.

If we take for the systems the two pencils $(C_r)$ and $(C_s)$ then $\mu_1 = \mu_2 = 1$ and (as ensues immediately from the principle of correspondence) $\nu_1 = 2(r-1)$, $\nu_2 = 2(s-1)$. So the order of the curve of contact is

$$2r + 2s - 3.$$

For the number of points of intersection of $l$ with $L$ remains

$$3rst - \alpha r - \beta s - \gamma t - 2t - (2r + 2s - 3) = 3(rst + 1) - 2(r + s + t) - (\alpha r + \beta s + \gamma t).$$

So we find:

*The locus $L$ of the pairs consisting of two movable points by which a curve of each of the pencils is possible is of order*

$$n = 3(rst + 1) - 2(r + s + t) - (\alpha r + \beta s + \gamma t);$$

*here $\alpha$ is the number of fixed points of intersection of the pencils $(C_s)$ and $(C_t)$, $\beta$ that of the pencils $(C_t)$ and $(C_r)$ and $\gamma$ that of $(C_r)$ and $(C_s)$.*

2. Whilst the preceding considerations remain accurate when of the basepoints of one and the same pencil some coincide, we shall suppose in the following that the pencils $(C_r)$, $(C_s)$ and $(C_t)$ have respectively $r^2$, $s^2$ and $t^2$ different basepoints, so that we can only allow the basepoints of one pencil to coincide in part with those of an other pencil. Then $\alpha$ is the number of common basepoints of the pencils $(C_s)$ and $(C_t)$ (which can however also belong to $(C_r)$), etc. If the pencils have no common basepoints ($\alpha = \beta = \gamma = 0$), the order of the locus becomes

$$3(rst + 1) - 2(r + s + t).$$

This is also in the case of common basepoints the order of the *total* locus as long as that is definite, i. e. as long as there are no basepoints common to the three pencils. If there is such a point, this furnishes together with an entirely arbitrary point a pair of points $PP$ through which a curve of each of the pencils is possible; of this pair of points however only one is movable. The locus proper however is still definite then.

A basepoint of the pencil $(C_r)$ only we call $A_r$, a common basepoint of the pencils $(C_s)$ and $(C_t)$ which is not a basepoint of the pencil $(C_r)$ we call $A_{st}$ and a common basepoint of the three pencils we call $A_{rst}$. If $\delta$ is the number of points $A_{rst}$ then the number of points $A_{st}$ amounts to $\alpha' = \alpha - \delta$, that of the points $A_{rt}$ to $\beta' = \beta - \delta$ and that of the points $A_{rs}$ to $\gamma' = \gamma - \delta$, whilst the number of points $A_r$ is equal to $r^2 - \beta' - \gamma' - \delta$, etc. By introduction of $\alpha'$, $\beta'$, $\gamma'$ and $\delta$ the order $n$ of this locus proper becomes

$$n = 3(rst + 1) - 2(r + s + t) - (\alpha' r + \beta' s + \gamma' t) - \delta(r + s + t).$$

From this we see that the order of the locus proper is lowered by $r$ on account of a common basepoint $A_{st}$. If there are no points $A_{rst}$ ($\delta = 0$) one can easily account for that lowering of order by noticing that from the total locus the $C_r$ passing through $A_{st}$ separates itself, as not belonging to the locus proper. The point $A_{st}$ furnishes namely together with an arbitrary point of that $C_r$ a pair of points satisfying the question; of which points however only the latter is movable [1]). Furthermore we see that a point $A_{rst}$ diminishes the order of $L$ by $r + s + t$, a fact one cannot account for by separation, the total locus becoming indefinite [2]).

3. The locus proper $L$ has in the basepoints of the three pencils multiple points, the multiplicities of which are easy to determine.

A basepoint $A_r$ of the pencil $(C_r)$ only is an $(st - \alpha - 1)$-fold point of $L$. In fact, the curves $C_s$ and $C_t$ passing through $A_r$ have, $A_r$ and the basepoints excepted, still $st - \alpha - 1$ points of intersection each of which combined with $A_r$ furnishes a pair of points satisfying the question. The tangents in $A_r$ to the curves $C_r$ passing through the $st - \alpha - 1$ mentioned points of intersection are the tangents of $L$ in the multiple point.

To determine the multiplicity of a point $A_{st}$ we remark that to obtain a pair of points satisfying the question and of which one of the movable points coincides with $A_{st}$, it is necessary for $C_r$ to pass through $A_{st}$ (by which it is determined), whilst $C_s$ and $C_t$ which always pass through $A_{st}$ must present a movable point of intersection in $A_{st}$, thus must touch each other in $A_{st}$. The question now rises: How often do two curves $C_s$ and $C_t$ touching each other in $A_{st}$ intersect each other again on the curve $C_r$ passing through $A_{st}$? To answer this question we introduce an arbitrary $C_s$ intersecting the above mentioned $C_r$ in $rs - \gamma - 1$ points differing from the basepoints. Through each of these points we allow a $C_t$ to pass which gives rise to a correspondence between the curves $C_s$ and $C_t$ (so likewise between its tangents in $A_{st}$) where $rs - \gamma - 1$ curves $C_t$ correspond to a $C_s$ and $rt - \beta - 1$ curves $C_s$ to a $C_t$. Thus for the curves $C_s$ and $C_t$ touching each other in $A_{st}$ it happens $(rs + rt - \beta - \gamma - 2)$

[1]) If $A_{st}$ counts for $\varepsilon$ fixed points of intersection of the curves $C_s$ and $C_t$, the $C_r$ passing through $A_{st}$ separates itself $\varepsilon$ times by which the degree of $L$ is lowered by $\varepsilon r$.

[2]) If $A_{rst}$ counts for $\varepsilon$ fixed points of intersection of $C_s$ and $C_t$, for $\zeta$ fixed points of intersection of $C_r$ and $C_t$ and for $\eta$ fixed points of intersection of $C_r$ and $C_s$, then $A_{rst}$ diminishes the order of $L$ by $\varepsilon r + \zeta s + \eta t$; this holds for a point $A_{st}$ too, but then we must regard $\zeta$ and $\eta$ as being zero.

times that $C_s$ and $C_r$ have besides $A_{st}$ another movable point of intersection, being at the same time movable point of intersection of $C_t$ and $C_r$. Here is included the case in which this second point of intersection coincides with $A_{st}$, thus where the curves $C_s$ and $C_t$ touch $C_r$ in $A_{st}$; then only one movable point of intersection of $C_s$ and $C_t$ still coincides with $A_{st}$, whilst there need be no other movable point of intersection lying on $C_r$, so that in this way we get no pair of points furnishing a branch of $L$ passing through $A_{st}$. So the point $A_{st}$ is an $(rs + rt - \beta - \gamma - 3)$-fold point of $L$.

To determine the multiplicity of a point $A_{rst}$ we have to consider how many times three curves $C_r$, $C_s$ and $C_t$ touching each other in $A_{rst}$ pass once more through a same point. To this end we consider an arbitrary $C_r$ and the $C_s$ which touches this $C_r$ in $A_{rst}$. Through each of the $rs - \gamma - 1$ points of intersection of these $C_r$ and $C_s$, differing from the basepoints, we allow a $C_t$ to pass. Then the question arises how many times this $C_t$ touches $C_r$ and $C_s$ in $A_{rst}$. Let us call $l_{rs}$ the common tangent in $A_{rst}$ of $C_r$ and $C_s$ and $l_t$ the tangent of $C_t$ in that point. To $l_{rs}$ correspond $rs-\gamma-1$ lines $l_t$. To find reversely how many lines $l_{rs}$ correspond to an arbitrary line $l_t$ we consider an arbitrary $C_r$ intersecting the $C$ determined by $l_t$ in $rt-\beta$ points differing from the basepoints. Through each of those points of intersection we imagine a $C_s$. If $l_r$ and $l$ are the tangents in $A_{rst}$ of $C_r$ and $C_s$ then $rt-\beta$ lines $l_s$ correspond to $l_r$ and $st-\alpha$ lines $l_r$ to $l_s$. The $rt + st - \alpha - \beta$ rays of coincidence indicate the lines $l_{rs}$ corresponding to $l_t$; to those rays of coincidence however belongs the line $l_t$ itself, which must not be counted, so that $rt+st-\alpha-\beta-1$ lines $l_{rs}$ corresponding to $l_t$ remain. So between the lines $l_{rs}$ and $l_t$ exists an $(rs-\gamma-1, rt + st-\alpha-\beta-1)$-correspondence.

The required lines $l_{rst}$ are indicated by the $st+tr+rs-(\alpha+\beta+\gamma)-2$ rays of coincidence of this correspondence of which however three must not be counted. When namely the contact in $A_{rst}$ of $C_r$ and $C_s$ becomes a contact of the second order one of the $rs-\gamma-1$ points of intersection differing in general from the basepoints of $C_r$ and $C_s$ coincides with $A_{rst}$, namely in the direction of $l_{rs}$. The $C_t$ passing through that point of intersection will touch $l_{rs}$ in $A_{rst}$ in other words $l_t$ coincides with $l_{rs}$. As however the curves $C_r$ and $C_s$, but not the curves $C_r$ and $C_t$, neither the curves $C_s$, $C_t$ have in $A_{rst}$ a contact of the second order we do not find in this way a pair of points satisfying the question. Now it happens three times with two pencils o curves with a common basepoint, between which a projective correspondence has been in such a way arranged that the

curves must touch each other in that basepoint, that this is a contact of the second order, so that from the number of rays of coincidence three must be subtracted to find that of the lines $l_{rst}$ wanted. From this ensues that the multiplicity of the point $A_{rst}$ amounts to $st + tr + rs - (\alpha + \beta + \gamma) - 5$.

So we find:

*A basepoint of the pencil* $(C_r)$ *only is a*

$$(st - \alpha - 1)\text{-}$$

*fold point of the locus proper* $L$. *A common basepoint of the pencils* $(C_s)$ *and* $(C_t)$ *which is not basepoint of* $(C_r)$ *is a*

$$(rs + rt - \beta - \gamma - 3)\text{-}$$

*fold point of* $L$ *and a common basepoint of the three pencils is a*

$$(st + tr + rs - \alpha - \beta - \gamma - 5)\text{-}$$

*fold point of* $L$ [1]).

4. With the help of the preceding the points of intersection of $L$ with an arbitrary curve of one of the pencils, e.g. a $C_r$, are easy to indicate. These are:

1. The $r^2 - \beta - \gamma + \delta$ points $A_r$ counting together for

$$(r^2 - \beta - \gamma + \delta)(st - \alpha - 1)$$

points of intersection.

2. The $\beta - \delta$ points $A_{rt}$ counting together for

$$(\beta - \delta)(sr + st - \alpha - \gamma - 3)$$

points of intersection.

3. The $\gamma - \delta$ points $A_{rs}$, giving

$$(\gamma - \delta)(tr + ts - \alpha - \beta - 3)$$

points of intersection.

4. The $\delta$ points $A_{rst}$, giving together

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[1]) If there are no points $A_{rst}$ ($\delta = 0$) and therefore the total locus is not indefinite, we can also ask after the multiplicities of the points $A_r$ and $A_{st}$ as points of the total locus. Now the improper part of the locus consists of $\alpha$ curves $C_r$, $\beta$ curves $C_s$ and $\gamma$ curves $C_t$. Of these pass through a point $A_r$ the $\alpha$ curves $C_r$ and through a point $A_{st}$ the $\beta$ curves $C_s$, the $\gamma$ curves $C_t$ and one of the curves $C_r$. From this ensues:

*A point* $A_r$ *is an* $(st - 1)$, *a point* $A_{st}$ *an* $(rs + rt - 2)$*-fold point of the total locus.*

So the multiplicity of $A_r$ as a point of the total locus is not changed by the coincidence of the basepoints, whilst the multiplicity of $A_{st}$ is equal to the sum of the multiplicities which this point would have if it were only basepoint of the pencil $(C_s)$ or only basepoint of the pencil $(C_t)$.

$$\delta(st + tr + rs - \alpha - \beta - \gamma - 5)$$

points of intersection.

5. The movable points of intersection of $L$ with $C_r$; these are those points of intersection which displace themselves when we choose another $C_r$. These are found as the pairs of common points of the simply infinite linear systems of pointgroups intersect on $C_r$ by the pencils $(C_s)$ and $(C_t)$. The number of these are found from the following theorem:

*If there are on a curve of genus $p$ two simply infinite linear systems of pointgroups consisting of $a$ and $b$ points, the number of common pairs of points of those systems is*

$$(a - 1)(b - 1) - p.$$

In our case $a = rs - \gamma$, $b = rt - \beta$ and (as $C_r$ is an arbitrary curve of the pencil $(C_r)$) $p = \frac{1}{2}(r-1)(r-2)$. For the number of pairs of common points we therefore find

$$(rs - \gamma - 1)(rt - \beta - 1) - \tfrac{1}{2}(r - 1)(r - 2),$$

*and for the number of movable points of intersection of $L$ and $C_r$:*

$$2(rs - \gamma - 1)(rt - \beta - 1) - (r - 1)(r - 2).$$

So the total number of points of intersection is:

$$r(3rst + 3 - 2r - 2s - 2t - \alpha r - \beta s - \gamma t),$$

in accordance with the formula we have found for the order of $L$.

5. The pairs of points $PP'$ through which a curve of each of the pencils is possible determine on $L$ an involutory (1,1)-correspondence; in the following we shall indicate $P$ and $P'$ as corresponding points of $L$.

If $P$ falls into a doublepoint of $L$ differing from the basepoints, then in general two different points $P'$ and $P''$ will correspond to $P$ according to our regarding $P$ as point of the one or of the other branch of $L$ passing through $P$. The curves of the pencils passing through $P$ now have two more common points $P'$ and $P''$, so that we get a triplet of points $P\,P'\,P''$, through which a curve of each of the pencils is possible.

It may however also happen that the points $P'$ and $P''$ coincide. In that case correspond to the two branches through $P$ two branches through $P'$, so that $P'$ is likewise doublepoint of $L$. The curves of the pencils passing through $P$ have now but one other common point $P'$, but now the particularity arises that $P$ or $P'$ can be displaced in two ways such that the other common point is retained. So $PP'$ is then to be regarded as a double corresponding pair of points.

If reversely we have a triplet of points $PP'P''$ lying on curves of each of the pencils, then $P$ is a doublepoint of $L$, for $P'$ as well as $P''$ corresponds to $P$, and so it must be possible to displace $P$ in such a way that the corresponding point describes a branch passing through $P'$ and in such a way that a branch passing through $P''$ is described. The curve $L$ has thus two branches $P1$ and $P2$ passing through $P$ to which the branches $P'1$ and $P'2$ correspond. Through the point $P'$ (which is of course likewise doublepoint of $L$ as well as $P''$) a second branch $P'3$ passes and through $P''$ a second branch $P''3$, which branches correspond mutually. If a point $Q$ describes the branch $P1$ the curves $C_r$, $C_s$, $C_t$ passing through $Q$ have a second common point describing the branch $P'1$, whilst a third common point $P''$ appears and again disappears when $Q$ passes the point $P$. This third common point displaces itself (along the branch $P''2$) when $Q$ describes the other branch passing through $P$, whilst then the common point coinciding with $P'$ appears and disappears.

Triplets of points $PP'P''$, and therefore doublepoints of $L$ differing from the basepoints, there will be as a triplet of points depends on 6 parameters and it is a 6-fold condition that a curve of each of the pencils must pass through it. So we have:

*The curve $L$ has doublepoints, differing from the basepoints of the pencils, belonging in triplets together and forming the triplets of points through which a curve of each of the pencils is possible. To one or other branch through a doublepoint of such a triplet corresponds a branch through the second resp. the third doublepoint of this triplet. Moreover $L$ can however have pairs of doublepoints indicating the double corresponding pairs of points. To the two branches through the doublepoint of such a pair correspond the branches through the other doublepoint of the pair.*

6. The number of coincidences of the correspondence between $P$ and $P'$ can be determined as follows. The points $P$ and $P'$ coincide if the curves $C_r$, $C_s$ and $C_t$ passing through $P$ have in $P$ the same tangent. Then $P$ must lie on the curve of contact $R_{rs}$ of the pencils $(C_r)$ and $(C_s)$ as well as on the curve of contact $R_{rt}$ of $(C_r)$ and $(C_t)$. The number of points of intersection of those curves of contact which are of order $2r + 2s - 3$ resp. $2r + 2t - 3$ amounts to

$$(2r + 2s - 3)(2r + 2t - 3).$$

Some of these points of intersection however do not lie on the third curve of contact $R_{st}$, and so they must not be counted. The curve $R_{rs}$ namely passes once through a basepoint $A_r$ or $A_s$ and three times through a common basepoint $A_{rs}$ or $A_{rst}$; in fact in a point of

$R_{rs}$ two movable points of intersection of $C_r$ and $C_s$, coincide so that the point $A_{rs}$ as a point of the curve of contact is found when $C_r$ and $C_s$ show in $A_{rs}$ a contact of the second orderwhich takes place three times. Further $R_{rs}$ passes through the doublepoints of the curves $C_r$ and $C_s$, of which the number for the pencil $(C_r)$ amounts to $3(r-1)^2$ and for the pencil $(C_s)$ to $3(s-1)^2$, which follows immediately from the order of the discriminant.

Each of the $r^2 - \beta' - \gamma' - \delta$ points $A_r$ is a simple point of intersection of $R_{rs}$ and $R_{rt}$ (simple, the tangents in $A_r$ to $R_{rs}$ and $R_{rt}$ being the tangents of the curves $C_s$ and $C_t$ passing through $A_r$, differing thus in general), but no point of $R_{st}$. Each of the $\alpha'$ points $A_{st}$ is a double point of intersection of $R_{rs}$ and $R_{rt}$, as those curves of contact in $A_{st}$ have a simple point with the same tangent, namely that of the $C_r$ passing through $A_{st}$; these points are also points of $R_{st}$, namely threefold ones. Each of the $\beta'$ points $A_{rt}$ is threefold point of intersection of $R_{rs}$ and $R_{rt}$ (it being simple point of $R_{rs}$ and threefold point of $R_{rt}$) and lies at the same time on $R_{st}$; the same holds for the $\gamma'$ points $A_{rs}$. Each of the $\delta$ points $A_{rst}$ which are common basepoints of the three pencils is 9-fold point of intersection of $R_{rs}$ and $R_{rt}$, being threefold point of each of those curves; moreover it is threefold point of $R_{st}$. Finally the $3(r-1)^2$ doublepoints of the pencil $(C_r)$ are simple points of intersection of $R_{rs}$ and $R_{rt}$, but not points of $R_{st}$; of the curves $C_r$, $C_s$ and $C_t$ passing through such a doublepoint $C_r$ has an improper contact with $C_s$ and with $C_t$, without however $C_s$ and $C_t$ touching each other.

From this we see that the curves of contact $R_{rs}$ and $R_{rt}$ have

$$r^2 - \beta' - \gamma' - \delta + 3(r-1)^2 = 4r^2 - 6r + 3 - \beta' - \gamma' - \delta$$

points of intersection which are not points of $R_{st}$, and so do not furnish coinciding points $P$, $P'$. Moreover $R_{rs}$ and $R_{rt}$ have

$$2\alpha' + 3\beta' + 3\gamma' + 9\delta$$

points of intersection coinciding with the common basepoints, which do fall on $R_{st}$, but which do not give any coinciding points $P$ and $P'$, as for this it is necessary that of three curves $C_r$, $C_s$ and $C_t$ passing through the same point each pair shows two movable points of intersection coinciding with that point. So for the number of coinciding points $P$ and $P'$ remains:

$$(2r + 2s - 3)(2r + 2t - 3) - (4r^2 - 6r + 3 - \beta' - \gamma' - \delta) - \\ - (2\alpha' + 3\beta' + 3\gamma' + 9\delta) = \\ = 4(st + tr + rs) - 6(r + s + t) + 6 - 2(\alpha' + \beta' + \gamma' + 4\delta).$$

So we find:

*It happens*

$$4(st + tr + rs) - 6(r + s + t) + 6 - 2(\alpha + \beta + \gamma + \delta)$$

*times that the two points $P$ and $P'$ through which a curve of each of the pencils is possible coincide.*

7. With the help of this result the class of the envelope of the lines connecting $P$ and $P'$ can easily be determined. To this end we have to count how many lines $PP'$ pass through an arbitrary point $S$. We find this number by regarding the correspondence between the rays $SP$ and $SP'$, which we call $l$ and $l'$. This is an involutory $(n, n)$-correspondence where $n$ represents the order of the locus $L$ of the points $P$ and $P'$; for on an arbitrary ray $l$ (or $l'$) lie $n$ points $P$ (or $P'$), to each of which one point $P'$ (or $P$) corresponds. So there are $2n$ rays of coincidence which can be furnished either on account of $PP'$ passing through $S$ or of $P$ and $P'$ coinciding.

So for the number of rays of coincidence where $PP'$ passes through $S$ we find:

$$2\{3(rst+1)-2(r+s+t)-(\alpha r+\beta s+\gamma t)\}-\{4(st+tr+rs)-$$
$$-6(r+s+t)+6-2(\alpha+\beta+\gamma+\delta)\}=6rst-4(st+tr+rs)+$$
$$+2(r+s+t)-2\alpha(r-1)-2\beta(s-1)-2\gamma(t-1)+2\delta.$$

These rays of coincidence however coincide in pairs. For if the line connecting the corresponding points $P_1$ and $P_1'$ passes through $S$, then to $P_1P_1'$ regarded as line $l$ correspond $n$ lines $l'$, two of which coincide with $P_1P_1'$, for if point $P$ of $l$ is taken in $P_1$ or in $P_1'$ the corresponding point $P'$ lies in $P_1'$ resp. $P_1$. Likewise to $P_1P_1$ regarded as line $l'$ correspond $n$ lines $l$, of which also two coincide with $P_1P_1'$, from which ensues that $P_1P_1'$ is a double ray of coincidence [1]). So to find the number of the lines $PP'$ passing through $S$, thus the class of the envelope, the above found number must still be divided by 2, so that we get:

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[1]) One can easily convince oneself of the accuracy of this conclusion by a representation of the correspondence between the rays $SP$ and $SP'$. To this end we regard the parameters of the lines $SP$ and $SP'$ as rectangular Cartesian coordinates $x$ and $y$ of a point which is the representation of those two lines. The curve of representation (which is symmetrical with respect to the line $y=x$ on account of the correspondence being involutory) indicates by its points of intersection with the line $y=x$ the rays of coincidence. If $B$ is the point of representation of the rays $l$ and $l'$ coinciding in $P_1P_1'$, the curve of representation is cut in two coinciding points $B$ by a line parallel to the $y$-axis as well as by a line parallel to the $x$-axis, on account of $P_1P_1'$ regarded as $l$ or $l'$ corresponding twice to itself regarded as $l'$ resp. $l$. So $B$ is doublepoint of the curve of representation, so that the line $y=x$ furnishes two points of intersection coinciding with $B$.

*The envelope of the lines connecting pairs of points, through which a curve of each of the pencils is possible, is of class*

$$3\,rst - 2\,(st+tr+rs) + (r+s+t) - \alpha(r-1) - \beta(s-1) - \gamma(t-1) + \delta =$$
$$= 3\,rst - 2\,(st+tr+rs) + (r+s+t) - \alpha'(r-1) - \beta'(s-1) - \gamma'(t-1) -$$
$$- \delta\,(r+s+t-4).$$

8. If the pencils have no common basepoints then the class of the envelope is $3\,rst - 2\,(st + tr + rs) + (r + s + t)$. By a common basepoint $A_{st}$ of the pencils $(C_s)$ and $(C_t)$ that class is lowered with $r - 1$. *This is because point $A_{st}$ has separated itself from the envelope $r$—1 times.* In fact, the curve $C_r$ passing through $A_{st}$ has separated itself from the locus of the points $P$ and $P'$. If we take $P$ arbitrarily on this $C_r$, the corresponding point $P'$ coincides with $A_{st}$. So an arbitrary line passing through $A_{st}$ is to be regarded $(r - 1)$ times as a line connecting $P$ and $P'$, as any of the $r - 1$ points of intersection with $C_r$ differing from $A_{st}$ may be chosen for $P$.

If the three pencils have a common basepoint $A_{rst}$ the total envelope of $PP'$ remains definite (in contrast to the total locus of $P$ and $P'$). It is true $P$ can be taken quite arbitrarily, but then $P'$ coincides with a point $A_{rst}$, so that the line $PP'$ passes through that point $A_{rst}$, and therefore is not quite arbitrary. As the class of the envelope proper is lowered by the point $A_{rst}$ with $r + s + t - 4$ it follows, *that $A_{rst}$ separates itself $(r + s + t - 4)$ times from the envelope.* As one of the points of the pair becomes entirely indefinite, that multiplicity is not easy to explain, as far as I can see.

**Physics.** — "*On a new empiric spectral formula.*" By E. E. Mogendorff. (Communicated by Prof. P. Zeeman).

By the fundamental investigations of Kayser and Runge and those of Rydberg the existence of spectral series was proved. The formulae of these physicists, however, give in general too great deviations for the first lines of a series. I have tried to improve the formula given by Rydberg:

$$n = A - \frac{N_0}{(m + a)^2}.$$

Particularly noteworthy in Rydberg's formula is the universal constant $N_0$. From Balmer's formula, which is included as a special case in Rydberg's formula, follows for hydrogen for the observation corrected to vacuo $N_0 = 109675$.

Assuming for a moment that the $N_0$ was also variable for the different series, I have calculated the constants $A$, $a$ and $N_0$ from three of the best observed curves. For $N_0$ the following values were found:

| | | |
|---|---|---|
| Principal Series | Lithium | 109996 |
| ,, ,, | Natrium | 107178 |
| ,, ,, | Potassium | 105638 |
| ,, ,, | Rubidium | 104723 |
| ,, ,, | Caesium | 104665 |
| 1st associated series | Hydrogen | 109704 |
| ,, ,, | Helium | 109703 |
| ,, ,, | Natrium | 110262 |
| ,, ,, | Potassium | 109081 |
| ,, ,, | Silver | 107162 |
| ,, ,, | Magnesium | 108695 |
| ,, ,, | Zinc | 107489 |
| ,, ,, | Oxygen | 110660 |
| Second ,, | Natrium | 107819 |
| ,, ,, | Magnesium | 105247 |
| ,, ,, | Calcium | 105702 |
| ,, ,, | Zinc | 105399 |
| ,, ,, | Aluminium | 105721 |

These values have been calculated from wave frequencies not corrected to vacuo.

As appears from these values $N_0$ is not absolutely constant. As KAYSER [1]) found in another way, we see, however, that relatively

[1]) KAYSER, Handbuch II. p. 553.

$N_0$ changes little from element to element [1]). The supposition lies at hand, that a constant of nature will occur in the rational formula. For the first associated series of Aluminium calculation gives a considerable deviation. Calculating from the first terms of this series we find $N_0 = 207620$ calculating from the middle lines $N_0 = 138032$, and from the lines with smaller $\lambda$ $N_0 = 125048$.

The first asssociated series of aluminium behaves therefore quite abnormally.

In RYDBERG's formula another function than $(m + a)^{-2}$ must be used to get a better harmony, specially with the first terms of a series.

In my thesis for the doctorate, which will shortly appear, I have examined the formula:

$$n = A - \frac{109675}{\left(m + a + \frac{b}{m}\right)^2},$$

in which $n$ represents the wave frequency reduced to vacuo, $A$, $a$ and $b$ are constants which are to be determined, $m$ passes through the series of the positive integers, starting with $m = 1$. In most cases with this formula a good agreement is obtained, also with the first lines of a series. The associated series converge pretty well to the same limit, while also the law of RYDBERG—SCHUSTER is satisfied in those cases where besides associated series, also a principal series is observed.

A spectral formula has also been proposed by RITZ [2]).

In my thesis for the doctorate I have adduced some objections to the formula of RITZ, as it gives rise to highly improbable combinations of lines. Moreover for the metals of the 2nd column of MENDELEJEFF's system his views are not at all in harmony with observation.

In the following tables the observed wavelength in Å. E. is given under $\lambda_w$, the limit of error of observation under F, the deviation according to the formula proposed by me under A, the deviation according to the formula of KAYSER and RUNGE under A. K. R. The mark * on the right above a wavelength indicates that these lines were used as a basis for the calculation of the constants A, $a$ and $b$.

The constants are calculated from the wave frequencies reduced to vacuo [3]).

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[1]) The $B$ in KAYSER and RUNGE's formula varies within considerably wider limits than the $N_0$ of RYDBERG's formula.

[2]) Ann. d. Phys. Bd. 12, 1903, p. 264. W. RITZ, Zur Theorie der Serienspectren.

[3]) Where it was possible, I have always taken these values from the "Index of Spectra" from MARSHALL WATTS.

*Lithium.*

Principal series: $A = 43480{,}13$ ; $a = +\,0{,}95182$ ; $b = +\,0{,}00722$
1st ass. series : $A = 28581{,}8$ ; $a = +\,1{,}998774$; $b = -\,0{,}000822$
2nd ,, : $A = 28581{,}8$ ; $a = +\,1{,}59872$ ; $b = -\,0{,}00321$
3rd ,, ; $A = 28581{,}8$ ; $a = +\,1{,}95085$ ; $b = +\,0{,}00404$

The associated series converge here evidently to one limit.

The difference of wave frequency between the limits of principal and associated series is $43480{,}13 - 21581{,}8 = 14898{,}33$. The wave frequency of the 1st line of the principal series is 14902,7. So the formula satisfies the law of RYDBERG-SCHUSTER pretty well.

PRINCIPAL SERIES.

| $m$ | $\lambda_W$ | F | K | A. K. R. |
|---|---|---|---|---|
| 1 | 6708,2 * | 0,20 | 0 | + 108 |
| 2 | 3232,77* | 0,03 | 0 | 0 |
| 3 | 2741,39 | 0,03 | — 0,06 | 0 |
| 4 | 2562,60* | 0,03 | 0 | 0 |
| 5 | 2475,13 | 0,10 | — 0,22 | — 0,2 |
| 6 | 2425,55 | 0,10 | — 0,18 | — 0,01 |
| 7 | 2394,54 | 0,20 | — 0,13 | + 0,30 |
| 8 | 2373,9 L. D. | ? | + 0,02 | + 0,75 |
| 9 | 2359,4 L. D. | ? | + 0,17 | + 1,18 |

FIRST ASSOCIATED SERIES.

| $m$ | $\lambda_W$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 6103,77* | 0,03 | 0 | 0 |
| 2 | 4602,37* | 0,10 | 0 | 0 |
| 3 | 4132,44 | 0,20 | — 0,11 | 0 |
| 4 | 3915,20* | 0,20 | 0 | — 0,20 |
| 5 | 3794,9 | 5,00 | + 0,09 | — 0,35 |
| 6 | 3718.9 | 5,00 | — 1,94 | — 2,25 |
| 7 | 3670,6 | 5,00 | — 1,06 | — 1,41 |

SECOND ASSOCIATED SERIES.

| $m$ | $\lambda_W$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 8127,0* S | 0,30 | 0 | — 65 |
| 2 | 4972,11 | 0,10 | — 0,13 | 0 |
| 3 | 4273,44* | 0,20 | 0 | 0 |
| 4 | 3985,94 | 0,20 | + 0,22 | 0 |
| 5 | 3838,30 | 3,00 | + 2,40 | — 0,2 |

THIRD ASSOCIATED SERIES.

| $m$ | $\lambda_W$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 6240,3* S | 0,40 | 0 | — |
| 2 | 4636,3* S | 0,40 | 0 | — |
| 3 | 4148,2 S | 1,00 | + 1,6 | — |
| 4 | 3921,8 E H | ? | — 0,88 | |

The capitals after the wavelengths denote the observers: L. D. Liveing and Dewar; S. Saunders and E. H. Exner and Haschek. Where no further indication is given, the observation has been made by Kayser and Runge.

*Natrium.*

Principal series (the lines of the doublets with greatest $\lambda$)

$$A = 41447{,}09; \quad a = 1{,}147615; \quad b = -0{,}031484$$

Principal series (lines of the doublets with smallest $\lambda$)

$$A = 41445{,}20; \quad a = 1{,}148883; \quad b = -0{,}031908.$$

For the calculation of the limit of the associated series Rydberg-Schuster's law has been used. With a view to the constant differences of wave frequencies of the doublets of the associated series, I have only carried out the calculation for the components with small wavelength.

For the 1st ass. series $A = 24491{,}1$; $a = 1{,}98259$; $b = +0{,}00639$
For the 2nd ass. series $A = 24491{,}1$; $a = 1{,}65160$; $b = -0{,}01056$

PRINCIPAL SERIES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 5896,16* | — | 0 | + 78 |
| 1 | 5890,19* | — | 0 | + 86 |
| 2 | 3303,07* | 0,03 | 0 | 0 |
| 2 | 3302,47* | 0,03 | 0 | 0 |
| 3 | 2852,91 | 0,05 | — 0,14 | 0 |
| 3 | 2852,91 | 0,05 | — 0,06 | 0 |
| 4 | 2680,46* | 0,10 | 0 | 0 |
| 4 | 2680,46* | 0,10 | 0 | 0 |
| 5 | 2593,98 | 0,10 | + 0,03 | + 0,03 |
| 5 | 2593,98 | 0,10 | — 0,02 | + 0,09 |
| 6 | 2543,85 L. D. | 0,10 | — 0,06 | + 0,10 |
| 6 | 2543,85 L. D. | 0,10 | — 0,14 | + 0,24 |
| 7 | 2512,23 L. D. | 0,20 | + 0,03 | + 0,50 |
| 7 | 2512,23 L. D. | 0,20 | — 0,10 | + 0,60 |

FIRST ASSOCIATED SERIES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 8184,33* L. | 0,2 | 0 | 0 |
| 2 | 5682,90 | 0,15 | 0,01 | 0 |
| 3 | 4979,30* | 0,20 | 0 | 0 |
| 4 | 4665,20 | 0,50 | — 0,13 | + 0,52 |
| 5 | 4494,30 | 1,00 | — 0,28 | + 0,50 |
| 6 | 4390,70 L. D. | ? | + 0,28 | + 1,30 |
| 7 | 4325,70 L. D. | ? | + 4,00 | + 1,76 |

SECOND ASSOCIATED SERIES.

| | | | | |
|---|---|---|---|---|
| 1 | 11404 | ? | + 1,00 | + 100,– |
| 2 | 6154,62* | 0,10 | 0 | 0 |
| 3 | 5149,19* | 0,10 | 0 | 0 |
| 4 | 4748,36 | 0,15 | + 0.12 | 0 |
| 5 | 4542,75 | 0,20 | + 0,65 | + 1,39 |
| 6 | 4420,20 L. D. | ? | + 0,02 | + 1,55 |
| 7 | 4343,70 | ? | + 2,00 | — 1,36 |

*Zinc.*

For this element I have calculated the formulae of the 1st and 2nd associated series for the components with the greatest wavelength of the triplets.

The limits are determined for the two series separately, for the first associated series the calculation gave 42876,25 and for the second associated series the limit appeared to be 42876,70. A very good agreement.

The formula gives as 1st line of the 1st associated series of Zinc the line 8024,05, which has not been observed. The 8th line of the first associated series 2409,22 has not been observed either. As 9th line of this series 2393,93 was calculated, which is in remarkably good harmony with the intense line 2393,88. As yet this line had not yet been fitted in the series. The great intensity of a curve in the root of the series is certainly strange; an investigation of the magnetic splitting might decide whether it is correct to range this line under the first associated series.

The formula for the 1st associated series is:

$$n = 42876{,}25 - \frac{109675}{\left(m + 0{,}909103 - \frac{0{,}007085}{m}\right)^2}$$

and for the 2nd associated series:

$$n = 42876{,}70 - \frac{109675}{\left(m + 1{,}286822 - \frac{0{,}058916}{m}\right)^2}$$

FIRST ASSOCIATED SERIES.

| $m$ | $\lambda_W$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | —— | — | — | — |
| 2 | 3345,13* | 0,03 | 0 | — 0,08 |
| 3 | 2801,00* | 0,03 | 0 | + 0,03 |
| 4 | 2608,65* | 0,05 | 0 | + 0,06 |
| 5 | 2516,00 | 0,20 | + 0,04 | — 0,11 |
| 6 | 2463,47 | 0,20 | — 0,14 | — 0,39 |
| 7 | 2430,74 | 0,30 | + 0,22 | + 9,00 |
| 8 | —— | — | — | — |
| 9 | 2393,88 | 0,05 | — 0,05 | — |

SECOND ASSOCIATED SERIES.

| $m$ | $\lambda_W$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 4810,71* | 0,03 | 0 | + 58 |
| 2 | 3072,19* | 0,05 | 0 | 0,00 |
| 3 | 2712,60* | 0,05 | 0 | + 0,02 |
| 4 | 2567,99* | 0,10 | + 0,11 | — 0,01 |
| | 2493,67 | 0,15 | + 0,12 | — 0,04 |
| 6 | 2449,76 | 0,25 | — 0,11 | — 0,20 |

*Thallium.*

The formula for the 1st associated series is:

$$n = 41466{,}4 - \frac{109675}{\left(m + 1{,}90141 - \frac{0{,}00366}{m}\right)}$$

for the satellites:

$$n = 41466{,}4 - \frac{109675}{\left(m + 1{,}88956 - \frac{0{,}00085}{m}\right)^2}$$

and for the second associated series:

$$n = 41466{,}4 - \frac{109675}{\left(m + 1{,}26516 - \frac{0{,}07108}{m}\right)^2}$$

The limit has been calculated from three lines of the 1st associated series; only two more lines were required of the satellites and of the 2nd associated series. So in this spectrum all the constantshavebeen calculated from 7 lines and 31 lines are very well represented by the formula.

FIRST ASSOCIATED SERIES.

| $m$ | $\lambda_W$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 3519,39* | 0,03 | 0 | — |
| 2 | 2918,43 | 0,03 | — 0,04 | — |
| 3 | 2709,33* | 0,03 | 0 | — |
| 4 | 2609,08 | 0,03 | + 0,04 | — |
| 5 | 2552,62* | 0,10 | 0 | — |
| 6 | 2517,50 | 0,10 | — 0,06 | — 0,34 |
| 7 | 2494,00 | 0,10 | — 0,03 | — 0,19 |
| 8 | 2477,58 | 0,10 | — 0,09 | + 0,06 |
| 9 | 2465,54 | 0,20 | — 0,17 | + 0,24 |
| 10 | 2456,53 | 0,20 | — 0,15 | + 0,47 |
| 11 | 2449,57 | 0,30 | — 0,17 | + 0,68 |
| 12 | 2444,00 | 0,30 | — 0,28 | + 0,79 |
| 13 | 2439,58 | 0,30 | — 0,24 | + 0,95 |

SATELLITES.

| $m$ | $\lambda_W$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 3529,58* | 0,03 | 0 | + 0,02 |
| 2 | 2921,63 | 0,03 | + 0,06 | — 0,07 |
| 3 | 2710,77* | 0,03 | 0 | + 0,13 |
| 4 | 2609,86 | 0,03 | — 0,03 | — 0,02 |
| 5 | 2553,07 | 0,10 | — 0,05 | — 0,12 |

SECOND ASSOCIATED SERIES.

| $m$ | $\lambda_W$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 5350,65* | 0,03 | 0 | — 168 |
| 2 | 3229,88* | 0,03 | 0 | — 21,7 |
| 3 | 2826,27 | 0,05 | — 0,05 | — 3,65 |
| 4 | 2665,67 | 0,05 | — 1,32 | — 1,69 |
| 5 | 2585,68 | 0,05 | — 0,16 | + 0,01 |
| 6 | 2538,27 | 0,10 | — 0,17 | + 0,04 |
| 7 | 2508,03 | 0,15 | — 0,14 | — 0,01 |
| 8 | 2487,57 | 0,20 | — 0,06 | + 0,08 |
| 9 | 2427,65 | 0,20 | — 0,34 | — 0,21 |
| 10 | 2462,01 | 0,30 | — 0,20 | — 0,03 |
| 11 | 2453,87 | 0,30 | — 0,17 | + 0,07 |
| 12 | 2447,59 | 0,30 | — 0,05 | + 0,22 |
| 13 | 2442,24 | 0,30 | — 0,37 | — 0,01 |

I shall just add a few words on the spectrum of Aluminium. None of the formulae given as yet represents the first associated series of this element at all satisfactorily; nor is a satisfactory result attained with my formula. In the beginning of this paper I have pointed out, that very deviating values for $N_0$ were calculated from three of the 1st lines of the series.

The formula runs:

$$n = 48287{,}9 - \frac{109675}{\left(m + 0{,}89436 + \frac{1{,}038060}{m}\right)^2}$$

The constants have been calculated from the lines 4, 5 and 6.

ALUMINIUM. FIRST ASSOCIATED SERIES.

| $m$ | $\lambda_W$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 3082,27 | 0,03 | — 268,82 | + 384,8 |
| 2 | 2568,08 | 0,03 | + 3,46 | + 53,5 |
| 3 | 2367,16 | 0,03 | + 2,52 | + 6,1 |
| 4 | 2263,83* | 0,10 | 0 | + 0,03 |
| 5 | 2204,73* | 0,10 | 0 | + 0,17 |
| 6 | 2168,87* | 0,10 | 0 | — 0,13 |
| 7 | 2145,48 | 0,20 | + 0,06 | — 0,31 |
| 8 | 2129,52 | 0,20 | + 0,11 | — 0,21 |
| 9 | 2118,58 | 0,20 | — 0,28 | + 0,44 |

The agreement with the first lines (1, 2 and 3), leaves much to be desired. The value of the constant $b$ is here 1,03806, greater than the value of $a$ in that formula; this does not occur with any of the other series.

With 4 constants, so with:

$$n = A - \frac{109675}{\left(m + a + \frac{b}{m} + \frac{c}{m^2}\right)^2}$$

a better result is most likely reached. When the constants $b$ and here probably also the $c$, are not small with respect to $a$, then the influence of those constants is very great, particularly for small values of $m$. The deviation for the first line of the above series (3082,27), however, is so great, that I doubt if this is really the first line of this series.

The behaviour of this Aluminium series is certainly peculiar, and a further investigation is desirable.

For the way in which the constants in the formula were calculated, and for the spectra of Potassium, Rubidium and Calcium, of Magnesium, Calcium, Cadmium and of Helium and Oxygen, I refer to my thesis for the doctorate, which will shortly be published.

**Astronomy.** — *"Mutual occultations and eclipses of the satellites of Jupiter in* 1908." By Prof. J. A. C. OUDEMANS.

## SECOND PART. — ECLIPSES.

(Communicated in the meeting of October 27, 1906).

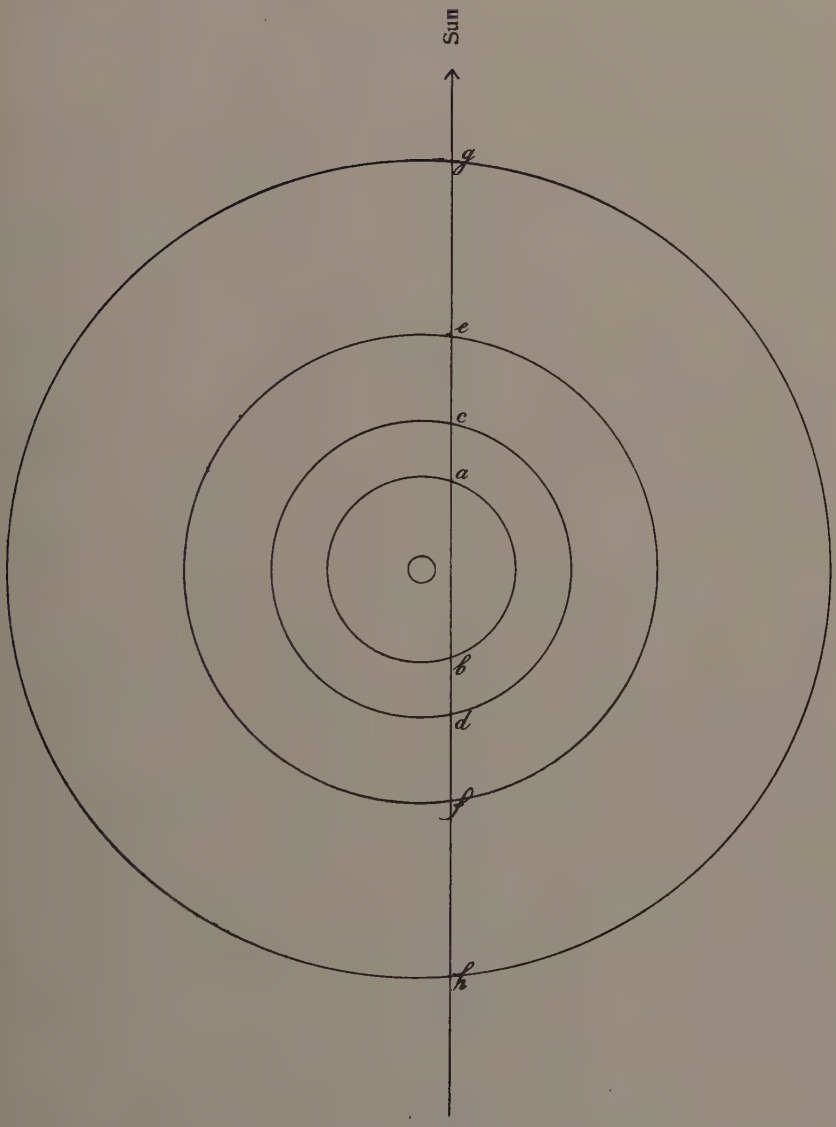
From occultations to eclipses there is but one step.

Between the two phenomena there is this difference that, as has been communicated on p. 305, the occultations have been observed more than once, but that of the eclipses of one satellite by another we have but one, incomplete account given in a private letter of Mr. STANLEY WILLIAMS dated 7 December 1905. In his letter to us he writes: "With regard to the heliocentric conjunctions there does "seem to be one observation of the rare phenomenon of the eclipse "of a satellite in the shadow of another one on record. It occurred "on the 14th August 1891 and was observed by Mr. J. COMAS at "VALLS in Spain and by the writer at HOVE. Mr. COMAS' observation "was published in the FRENCH periodical L'Astronomie, 1891, p. 397 "(read 398) 1). The following is an account of my observation. No "particulars of this have hitherto been published."

" "1891 Aug. 14. $6^1/_2$ inch reflector, power 225. Definition good, " "but interruptions from cloud. Satellite I. transitted on the S. Equa- " "torial belt, (N. component). *Immediately* on its entering the disc " "it became lost to view. At $11^h49^m$ a minute dark spot was seen " "about in the position which the satellite should have then occupied. " "The shadows of satellites I. and II. were confounded together at " "this time, there seeming to be one very large, slightly oval, black " "spot. At $11^h59^m$ the two shadows were seen neatly separated, " "thus, The preceding shadow must be that of II., the follow- " "ing and *much smaller* one that of I.. At $12^h10^m$ satellite I. was " "certainly visible as a dark spot, much smaller than the shadow " "of either satellite. It had moved with respect to the shoulder of " "the Red Spot Hollow, so that there could be no doubt of its " "identity. It is on the north band of the north (south) equatorial " "belt 2). Satellite I [this should evidently be II.] shines brightly " "on the disc near the limb. Definition good, but much thin cloud " "about." "

"The foregoing is an almost literal transcript from my observation "book. I take it that when satellite I. entered on the disc of Jupiter, "it was already partly eclipsed by the shadow of II., so that it "became lost to view immediately, instead of shining, as usual, for

J. A. C. OUDEMANS. "Mutual occultations and eclipses of the satellites of Jupiter in 1908." Second part: Eclipses.




Scale $\frac{1}{30\,168\,000\,000}$. On this scale the sun's diameter is 0.24 meter and its distance 25.783 m.

"some time as a brillant disc. Also that the minute dark spot seen "at $11^h49^m$ was produced by the portion of the shadow of II., then "projected on I. Also that the small size of the following shadow "spot at $11^h59^m$ was due to a part only of the shadow of II. being "projected on the disc of Jupiter, the other part of this shadow "having been intercepted by satellite I. 3)

— — — "But combining Mr. COMAS' observation with my own, "there can be no doubt but that satellite I. was actually partially "eclipsed by the shadow of II. on the night of August 14, 1891. "So far as I am aware, this is the only indubitable instance of one "satellite being eclipsed by the shadow of another." . . . . .

"P.S. The above times are Greenwich mean times. The Nautical "Almanac time for the transit ingress of satellite I. is $11^h33^m$." 4).

Before proceeding to the computation of epochs of such heliocentric conjunctions we have investigated to what extent generally eclipses of one satellite by the shadow of another are possible. That they may occur is proved by the shadows of the satellites on Jupiter itself. The question however is: 1st whether the shadows of the foremost satellite reaches that of the more distant one in *every* heliocentric conjunction and 2nd whether the occurrence of total eclipses is possible in any case. In order to find an answer to these questions we assume that the orbits all lie in a single plane which, being prolonged, passes through the centre of the sun. We further imagine a line in the plane of the orbits starting from the sun and passing Jupiter at a distance equal to its radius, the distance from the centre thus being equal to its diameter (see Plate I). This line cuts the orbits of the four satellites each in two points. Beginning with the point nearest the sun we shall call these points $g$, $e$, $c$, $a$, $b$, $d$, $f$ and $h$. For clearness, sake the figure is given below (Plate I).

Now suppose that I is placed either at $a$ or at $b$. In both cases the other satellites will be involved in its shadow cone as soon as they come: $II_f$ at $d$, $III_f$ at $f$ and $IV_f$ at $h$.

The points of intersection with the orbit of II are $c$ and $d$. If $II_n$ is at $c$ then $I_n$ may be eclipsed in $a$ but also $I_f$ in $b$; $III_f$ at $f$ and $IV_f$ at $h$.

But if $II_f$ is in $d$ then only $III_f$ and $IV_f$ can be eclipsed, the former at $f$ and the latter at $h$.

The points of intersection with the orbit of III are $e$ and $f$. If III is at $e$ there is the possibility of an eclipse for $II_n$ at $c$, $I_n$ at $a$, $I_f$ at $b$, $II_f$ at $d$ and $IV_f$ at $h$. If on the other hand it is in $f$ there is such a possibility only for $IV_f$ at $h$.

It is evident that IV can only cause the eclipse of another satel-

lite if it is at the position $g$, one of the other three satellites being then at one of the points of intersection already mentioned.

Each of the satellites might thus produce six different eclipses; if however we compute the radii of the umbra for the positions of the other satellites we are led to a negative value in some of the cases. This means of course that the vertex of the cone of the umbra does not reach the other satellite.

If for the radii of the satellites we adopt the values mentioned in the first part of this communication, diminished however by the amount of the irradiation, it appears that a total eclipse is only possible in two cases. $III_n$ may cause a total eclipse of $II_n$ and $I_n$; $I_f$ may nearly produce such an eclipse of $II_f$. If the shadow does not reach the other satellite then an inhabitant of the latter would see an annular eclipse of the Sun.

This case presents itself

for the shadow of $I_n$ in respect to $IV_f$.
„ $II_n$ „ „ „ $III_f$ and $IV_f$,
„ $III_n$ „ „ „ $IV_f$,
„ $IV_n$ „ „ „ $II_f$ and $III_f$.

In the fifteen remaining cases there may be a partial eclipse.

It need hardly be said that this case can only present itself if, at the time of heliocentric conjunction, the difference of the heliocentric latitudes $(y'—y)$, is smaller than the sum of the radii. In computing however the occultations observed by Messrs FAUTH and NIJLAND it appeared that this difference in latitude, according to the tables of DAMOISEAU, is sometimes slightly greater. The latitudes found by these tables are therefore not entirely trustworthy. For this reason we included *all* the heliocentric conjunctions between 1 April and 20 May 1908 (both dates inclusive).

The preparation for the computation, *viz* the drawing of the orbits of the satellites is the same as for the computation of the geocentric conjunctions (see 1st part). First however the epochs of the heliocentric superior conjunctions must be derived from the epochs of the *geocentric* superior conjunctions taken from the Nautical Almanac by the aid of the hourly motions of the satellites and of the angle $G$, i.e. the angle Earth—Jupiter—Sun. Furthermore, the jovicentric mean longitudes should be corrected for their equations and perturbations and diminished by S. i. e. the heliocentric longitude of Jupiter, instead of by S—G which is its geocentric longitude.

Of the arguments N°. 3 need not be computed; for this argument only serves, combined with 1, for the computation of the jovicentric

latitude of the Earth, which need not be known in the present case. The number of columns in our tables will thus be found to be diminished by one for each of the satellites.

Our results are contained in the annexed table. Between 1 April and 20 May we found 81 heliocentric conjunctions; the last column but one, $(y-y')$, shows that in a very great number of the cases an eclipse is possible.

(1) The account of Mr. José Comas is as follows:

*Ombres de deux satellites de Jupiter et éclipse.* — Dans la nuit du 14 août, j'ai observé un phénomène bien rare: la coïncidence partielle, sur Jupiter, des ombres de ses deux premiers satellites, et par suite l'éclipse de Soleil pour le satellite I produit par le satellite II.

A $11^h$ (temps de Barcelone) [1]) l'ombre du satellite II est entrée sur la planète. Près du bord, elle n'était pas noire, mais d'un gris rougeâtre. Comme l'image était fort agitée, j'ai cessé d'observer, mais je suis retourné à l'observation vers $11^h37^m$ pour observer l'immersion du premier satellite, qui a eu lieu à $11^h42^m$ (grossissement 100 fois; lunette de 4 pouces). J'ai été surpris de voir disparaître Io [2]) à son entrée sur le disque, ne se détachant pas en blanc, quoiqu'il se projetât sur la bande foncée équatoriale australe.

A $11^h52^m$, avec des images plus tranquilles et un grossissement de 160, je remarquai que l'ombre complètement noire que l'on voyait était allongée dans une direction un peu inclinée vers la droite, relativement à l'axe de Jupiter. La phase maxima de l'éclipse du satellite I était déjà passée de quelques minutes. A $11^h56^m$ je pris le petit dessin que j'ai l'honneur de vous adresser; les deux ombres se touchaient encore [3]). Aussitôt elles se séparèrent et, quoique je n'aie pas pu noter l'instant du dernier contact, je crois être assez près de la vérité, en disant qu'il s'est effectué vers $11^h58^m$.

L'empiètement d'une ombre sur l'autre pourrait être de la troisième

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[1]) Barcelone is 2°10′ East of Greenwich; mean time at Barcelone is therefore $8^m40^s$ later than of Greenwich.

[2]) Since a few years the Nautical Almanac mentions the names of the Satellites of Jupiter proposed by Simon Marius: Io, Europa, Ganymedes and Callisto.

[3]) This drawing shows, as seen in an inverting telescope, the right hand (following) part of the well know Red spot in the Southern Hemisphere of Jupiter. Below it, at some distance, a dark band and still further two dark shadows each 4 mm. in diameter, which are not yet separated. The common chord is 2,5 mm. in length; the total length of the two shadows together 7,2mm. The line connecting the centres makes an angle of 40° with the vertical. Meanwhile the motion of the two shadows must have been nearly horizontal.

partie du diamètre. Dans cette supposition la distance minima des centres des deux ombres a dû avoir lieu vers $11^h47^m$ et le premier contact vers $11^h37^m$. Le premier satellite pénétra dans le disque de la planète à $11^h42^m$, comme j'ai dit plus haut, donc l'éclipse a commencé quand le satellite se projetait encore dans l'espace, cinq minutes avant l'immersion.

L'invisibilité de l'ombre d'Europe sur Io peut s'expliquer par la mauvaise qualité des images. Toutefois, la pénombre et l'ombre du II satellite ont été suffisantes pour diminuer notablement l'éclat du premier.

(2) The meaning evidently is that, as seen in an *inverting* telescope the dark spot seemed to be situated on the North band of the North belt, but that in reality it was on the South band of the South belt. It is well known that the so-called *Red spot* is there situated.

(3) The author does not refer here to the visibility of a shadow of II on I. This may be explained, in my opinion, by irradiation and diffraction.

(4) According to the tables of Damoiseau, second part, the time of the heliocentric conjunction of the two satellites is $23^h45^m$ civil time Paris = $11^h36^m$ Greenwich. In the Nautical Almanac of 1891 we find the following data for 14 August:

| | | | | | | |
|---|---|---|---|---|---|---|
| II | Shadow. | Ingress | $10^h51^m$ | M. | T. | Grw. |
| I | „ | „ | 10 59 | „ | „ | „ |
| I | Transit | „ | 11 33 | „ | „ | „ |
| II | „ | „ | 11 58 | „ | „ | „ |
| I | Shadow. | Egress | 13 18 | „ | „ | „ |
| II | „ | „ | 13 45 | „ | „ | „ |
| I | Transit | „ | 13 51 | „ | „ | „ |
| II | „ | „ | 14 49 | „ | „ | „ |

If from the 1st, 2nd, 5th and 6th line we compute the time at which the shadows must coincide we get $11^h31^m$. This result differs by $5^m$ from that found just now. We have to consider, however, that the two satellites went the same way, and that their relative motion in five minutes, consequently also that of their shadows, was very minute.

Mr. Stanley Williams seems not to have perceived a shadow before $11^h49^m$ M. T. Greenwich; Mr. Comas already saw an oblong shadow at $11^h43^m20^s$ M. T. Greenwich. For the rest Mr. Stanley

WILLIAMS makes the shadow of II larger than that of I whereas in the estimation of Mr. COMAS they were equal. It seems hardly doubtful but the English observer must be right.

(5) In 1901 SEE repeatedly measured the diameters of the satellites of Jupiter at the 26 inch telescope of Washington. He made use of the filar micrometer but took a special care to eliminate the systematic errors peculiar to this instrument (*Vid.* Astron. Nachr. N°. 3764, 21 Jan. 1902. The communication of SEE is dated 19 Oct. 1901). During the months May—August (both inclusive) of the year 1901 he measured the diameters in the night. He was then much troubled by the undulation of the limbs caused by the unsteadiness of the air. Afterwards in the months of September and October of the same year he observed a little before and a little after sunset. Artificial illumination was then not needed; and the satellites appeared as quiet discs. Moreover the field and the satellites were coloured greenish yellow by a screen filled with protochloride of copper and picric acid. The results for the diameters turned out to be smaller in every case than those formerly found. The difference was attributed to irradiation.

The results, reduced to the mean distance of Jupiter to the sun (5,2028), are as follows.

| Satellite | At night | In daytime | Difference, attributed to irradiation |
|---|---|---|---|
| I | 1″,077 ± 0″018 | 0″834 ± 0″,006 | 0″243 ± 0″019 |
| II | 0 ,976 ± 0 ,043 | 0 ,747 ± 0 ,007 | 0 ,229 ± 0 ,0435 |
| III | 1 ,604 ± 0 ,038 | 1 ,265 ± 0 ,009 | 0 ,339 ± 0 ,039 |
| IV | 1 ,441 ± 0 ,018 | 1 ,169 ± 0 ,006 | 0 ,372 ± 0 ,019 |

It is remarkable that the brightest satellite, III, shows also the strongest irradiation. If however we consider the difference insufficiently established, and if therefore we combine the several results obtained for the irradiation, duly taking into account the weights corresponding to the probable errors, we get

$$\text{Irradiation} = 0'',264 \pm 0'',012.$$

This is the irradiation for the whole diameter and we thus get 0″,132 for each of the limbs. This number however holds only for the telescope at Washington for which, owing to its great aperture, the diffraction must be exceedingly small.

It seems worth while to call attention to the differences between the diameters found by the same observer in 1900 and 1901.

| | 1900 | 1901 | 1901—1900 |
|---|---|---|---|
| I | 0″672 ± 0″098 | 0″,834 ± 0″,006 | + 0″162 |
| II | 0,624 ± 0,078 | 0 ,747 ± 0 ,007 | + 0,121 |
| III | 1,361 ± 0,103 | 1 ,265 ± 0 ,009 | — 0,096 |
| IV | 1,277 ± 0,083 | 1 ,169 ± 0 ,006 | — 0,108 |

STONE, at Oxford, once told me that AIRY, in a conversation on the determination of declinations at the meridian circle, remarked to him: "I assure you, STONE, a second is a very small thing".

If we consider the differences just adduced between the results obtained by a single observer in two consecutive years we are led to conclude that, for micrometer observations, even now "a tenth of a second is an exceedingly small thing".

Appendix. *In how far are the tables of* DAMOISEAU *still reliable?*

In the first part of this paper, pages 319 and 321, we explained why we felt ourselves justified in using the tables of DAMOISEAU for these computations in advance. We may now add that we also investigated the differences of the eclipses, as observed in some recent years at different observatories, from these tables, or rather from the epochs given by the Nautical Almanac. In these investigations we have been assisted by Mr. KRESS, amanuensis at the Observatory of Utrecht, who has carefully searched some volumes of the *Astronomische Nachrichten* and of the *Monthly Notices* for the time of "disappearance and reappearance" of each satellite. He has further combined these times, reduced them to the meridian of Greenwich, and has then compared them with the data of the Nautical Almanac. In order to simplify, we requested him to note only the observation of the last light seen at disappearance and the first light at reappearance[1]). We intended to extend our investigation from 1894 to 1905

[1]) DELAMBRE in the introduction to his tables, does not state explicitly the *precise* instant to which his tables refer but from some passages we may conclude that he also means the instant *as here defined*. So for instance on page LIII where he says: "*Les demi-dureées ont été un peu diminuées, pour les rapprocher des observations qu'on a faites depuis la découverte des lunettes achromatiques*".

That LAPLACE also takes it for granted that such is his real meaning, appears from Ch. VIII, 8th book of the Mécanique Céleste.

or 1906, but after having completed some four years there seemed reason to think that there was hardly need for further information. The general result arrived at was, that the tables were still sufficiently accurate for our purpose, which was no other than to prepare astronomers for the observation of the mutual occultations and eclipses of the satellites.

Now that the work is finished we will not suppress its results though it cannot at all claim to be complete. It never was our intention to make it so, and the journals appearing in France, in America etc. have not been searched.

The following observatories have contributed to our investigation.

| | Aperture of the telescopes in m.m. |
|---|---|
| Greenwich . . . . . | 102, 170, 254, 714. |
| Utrecht . . . . . | 260 |
| Uccle . . . . . | 150 |
| Jena (WINKLER) . . . . . | 162 |
| Halifax (GLEDHILL) . . . . . | 237 |
| Pola . . . . . | 162 |
| Christiania . . . . . | 74, 190 |
| Kasan . . . . . | 66, 81, 84, 96, 244 |
| Göttingen . . . . . | 161 |
| Windsor (Tebbutt) near Adelaïde | 203 |
| Lyon (a single observation) | 2 |

At Greenwich, Christiania and Kasan the eclipses have been often observed by two or more astronomers using telescopes of different aperture. In such cases we have only taken into account the instant observed by means of the telescope of largest aperture. As a rule the observer at this telescope could follow the satellite longer at "disappearance" and he would pick it up earlier at "reappearance". There are however a few exceptions to the rule.

For the eclipses observed during the period of a single opposition of Jupiter the corrections to the data of the Nautical Almanac in no case showed a regular progression. They fluctuated on both sides of the mean in such a way that there could be no objection to adopting their arithmetical mean, a proceeding which still would be perfectly justified, even if there had been a *regularly* increasing or decreasing progression. No further attention was paid to the differences in the aperture of the telescopes. If these apertures exceed a certain amount, for instance 150 mm. we find, theoretically as

well as practically that the differences due to the varying apertures are very small.

The results arrived at are as follows:

Corrections to the epochs given in the Nautical Almanac for the eclipses of Jupiter's satellites.

| Opposition. | Mean Corr. N.A. Disapp. | Number. | Mean error. | Mean Corr. N.A. Reapp. | Number. | Mean error. | $\frac{1}{2}(D+R)$ | Mean error. |
|---|---|---|---|---|---|---|---|---|
| | | | | I. | | | | |
| 1894/95 | + 37s | 3 | ±14 | — 18s | 25 | ± 4s | + 9s.5 | ± 7s |
| 1895/96 | + 30 | 9 | 8 | 0 | 32 | 4 | + 15 | 4s |
| 1897 | — 19s | 2 | 18 | — 5 | 12 | 6 | — 12 | 9s |
| 1898 | + 11 | 15 | 6 | + 7 | 13 | 6 | + 6 | 4 |
| | | | | II. | | | | |
| 9418 | — 78 | 2 | ±32 | 0 | 7 | ±11s | — 39s | ±17s |
| 1894/95 | + 52 | 4 | 22s | — 42 | 15 | 7s | + 5 | 12 |
| 1895/96 | + 73 | 6 | 18 | — 4 | 19 | 6s | + 34 | 10 |
| 1897 | — 72 | 3 | 26 | + 11 | 10 | 9 | — 30 | 14 |
| 1898 | — 36 | 5 | 20 | — 15 | 9 | 9s | — 26 | 11 |
| | | | | III. | | | | |
| 1894 | +151s | 3 | ±22s | —242s | 3 | ±38s | — 45s | ±25s |
| 1895 | +101 | 4 | 19 | —127 | 4 | 33 | — 13 | 11s |
| 1895/96 | + 87 | 9 | 13 | — 50 | 9 | 22 | + 19 | 11s |
| 1897 | +181 | 4 | 19 | + 37 | 9 | 22 | +109 | 14s |
| 1898 | +266 | 4 | 19 | + 10 | 1 | 66 | +138 | 34 |
| 1899 | +361 | 3 | 22 | —126 | 4 | 33 | +118 | 20 |
| | | | | IV. | | | | |
| 1895 | + 21m45s | 3 | — | — 17m 9s | 2 | | +138s | |
| . . . | . . . . . | . . | . . | . . . . . | . . | . . | . . | . . |
| 1895/96 | + 3 49 | 10 | ±25 | — 3 17 | 7 | ±22s | + 16 | ±17s |
| 1897 | — 0 2 | 2 | ±57 | + 1 16 | 1 | 60 | + 37 | —41 |

Average mean error of a single observation.

| | Disappearance | Reappearance | Mean | Delambre *) Introd. p. LIV |
|---|---|---|---|---|
| I | $\pm$ 25s | $\pm$ 20s | $\pm$ 22s5 | 17s5 |
| II | 45 | 29 | 37 | |
| III | 37 | 66 | 51,5 | {88,5 / 72,5 rejecting the observations deviating more than 3 minutes). |
| IV | 80 | 60 | 70 | |

According to these numbers the complaints about the increased inaccuracy of the tables of DAMOISEAU seem rather exaggerated, at least for the first and second satellites.

Taking into account the mean errors contained in the last column we get the most probable correction at the epoch 1894—98

for I $+ 8^s,0$ with a mean error of $\pm 2^s,6$
similarly „ II $- 3\ ,8$ „ „ „ „ „ $\pm 5\ ,4$.

Both corrections can hardly be vouched for.

For III the case stands otherwise. It is true, the subtractive correction at the reappearances as well as the additive one at the disappearances may be attributable to the use of more powerful telescopes; still there seems to be a progression in the numbers of the last column but one, which calls for a more exhaustive investigation.

In regard to IV, we found great corrections for the year 1895. After some years in which this satellite had not been eclipsed, owing to the fact that at the opposition it passed to the north of the shadow cone of Jupiter, there began a new period of eclipses in this year. In such a case the satellite travels high above the plane of the orbit of Jupiter, and describes only a small chord in the shadow. The consequence is that any small error in the latitude appears strongly magnified in the duration of the eclipse. The observations of Mr. WINKLER at Jena and of the observer at the observatory at Uccle near Brussels, of 8 March 1895 are very suggestive in this regard. The corrections were found to be:

| | Jena. | Brussels. | Mean |
|---|---|---|---|
| at disappearance | $+ 19^m 48^s$ | $+ 21^m 58^s$ | $+ 20^m 53^s$ |
| at reappearance | $- 19\ 36$ | $- 18\ 33$ | $- 19\ 4,5$ |

which shows that it is not the mean longitude of this satellite which is mainly in error.

*) DELAMBRE gives mean *differences*; we have multiplied his numbers by $1^1/_4$ in order to get mean *errors*.

The explanation of these extravagant differences must rather be sought, either in a correction needed by the longitude of the node of the satellite's orbit or in the adopted flattening of Jupiter. It is also possible that for suchlike eclipses the diminution of light is *very slow.*

For the rest, according to the Nautical Almanac, this eclipse would be the fourth after the long period in which no eclipse of this satellite occurred. The data, on pages 450, 452, 454 are as follows:

| 1895 | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 17 Jan$^{y}$. | D. | $1^h$ | $36^m$ | $16^s$ | M. T. Gr., R. | $2^h$ | $8^m$ | $17^s$, | duration | | $32^m$ | $1^s$ |
| 2 Feb$^{y}$. | „ | 19 | 26 | 12 | „ „ „ „ | 20 | 36 | 58, | „ | $1^h$ | 10 | 46 |
| 19 „ | „ | 13 | 24 | 6 | „ „ „ „ | 14 | 59 | 3, | „ | 1 | 34 | 57 |
| 8 March | „ | 7 | 24 | 14 | „ „ „ „ | 9 | 18 | 28, | „ | 1 | 54 | 14. |

Only, according to Scott-Hansen, who, on the North-Polar expedition of Nansen, was in charge of the astronomical observations, the satellite has not been eclipsed at all on the 17$^{th}$ of January [1]).

On the 2$^{nd}$ February 1895 too an eclipse of IV was not observed; (I cannot now call to mind where I saw this negative observation). On the 19$^{th}$ February, however, an observer at Greenwich, using the Sheephanks equatorial, aperture 120 mm., got a correction of $+ 23^m30^s$, for the disappearance of IV. This agrees quite well with the preceding results, obtained at Uccle and at Jena on 8$^{th}$ of March.

If we adopt the mean result of the observations at Brussels and at Jena, the duration of the eclipse on that day was

$$1^h44^m14^s - 39^m57^s,5 = 1^h14^m16^s,5,$$

The number might be of some use for the correction of the elements of IV.

The difference here found cannot be attributed to a too small value of the adopted flattening, for Damoiseau's value $\frac{1}{13,49}$ exceeds already that found by direct measurement by most observers. Taking into account however the results obtained by De Sitter, as communicated at the meeting of the Section (Proceedings Vol. VIII p. 777), it appears that the longitude of the ascending node of the 4$^{th}$ satellite must be increased by about $+ 10°$, whereas for the inclination on

[1]) The Norwegian North Polar Expedition 1894 – 1896. Scientific Results, edited by Fridtjof Nansen. VI. Astronomical Observations, arranged and reduced under the supervision of H. Geelmuyden, p. XXIV.

the fixed plane is found the value $= 0°,2504 = 15'\,2''4$, which exceeds DAMOISEAU's inclination only by somewhat less than a minute.

The remaining eclipses of IV in 1895 and the two following years do not show any extraordinary divergencies.

Now, as in 1908 the eclipses of the satellites will be nearly central, as may be gathered from the drawings in the Nautical Almanac accompanying the table of these phenomena, there is no need to fear that such great divergencies will occur for IV in that year.

Our result therefore is that the Nautical Almanac, which is based on the tables of DAMOISEAU (taking into account only a few necessary corrections), may be considered sufficient for preparing ourselves for the coming observations. The only exception would be for an early eclipse of IV after a period in which it is not eclipsed at all.

*Utrecht,* 23 November 1906.

# RESULTS.

## Mutual heliocentric conjunctions of the satellites in April and May 1908.

A.A. = Ann Arbor; Fl. = Flagstaff; H.K. = Hong Kong; La Pl. = La Plata; P. = Perth; Tac. = Tacubaja; To. = Tokio; We. = Wellington; Wi. = Windsor.

| No. | Mean time at Greenwich | $n$ = near, $f$ = far: Eclipsed satellite | $n$ = near, $f$ = far: Eclipsing satellite | $x=x'$ | Eclipsed satellite $y$ | Eclipsing satellite $y'$ | $y-y'$ | Visible at |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 April 4h 8m | I$_f$ | II$_n$ | +5r70 | −0r30 | −0r25$^5$ | −0r04$^5$ | Kas., Taschk., Madras, HK., Perth. |
| 2 | 2 » 18 3 | I$_f$ | III$_n$ | +3,21 | −0,16 | −0,20 | +0,04 | Lick, Fl., Tac., AA., Harvard. |
| 3 | 3 » 4 15 | II$_f$ | III$_n$ | −2,49 | +0,08 | +0,10 | −0,02 | Kas., Taschk., Madr., HK., Perth, To. |
| 4 | 3 » 9 51 | II$_f$ | I$_n$ | +1,50 | −0,10$^5$ | −0,09 | −0,01$^5$ | Grw., Pulk., Kas., Taschk., La Pl., Rio. |
| 5 | 3 » 11 10 | IV$_f$ | III$_n$ | −6,19$^5$ | +0,40 | +0,32 | +0,08 | Grw., Pulk., Kasan, La Pl., Rio. |
| 6 | 3 » 16 26 | IV$_f$ | I$_n$ | −4,03 | +0,30 | +0,19 | +0,11 | Lick, Fl., Tac., AA., Harv., La Pl. |
| 7 | 4 » 16 52 | IV$_f$ | II$_n$ | +6,03 | −0.20 | −0,27 | +0,07 | Lick, Fl., Tac., AA., Harvard. |
| 8 | 4 » 17 21 | I$_f$ | II$_n$ | +5,75 | −0,31 | −0,25 | −0,06 | Lick, Fl., Tac., AA., Harvard. |
| 9 | 5 » 19 56 | III$_f$ | II$_n$ | −9,24 | +0,54 | +0,30 | +0,24 | Wi., We., Lick, Fl., Tac., AA. |
| 10 | 6 » 20 12 | III$_f$ | I$_n$ | +3,61 | −0,18$^5$ | −0,21 | +0,02$^5$ | Wi., We., Lick, Fl., Tac., AA. |
| 11 | 6 » 22 58 | II$_f$ | I$_n$ | +1,37 | −0,10 | −0,09 | −0,01 | Perth, Tokio, Wi., We. |
| 12 | 8 » 6 31 | I$_{e.e.}$ | II$_n$ | +5,82 | −0,31 | −0,25 | −0,06 | Bresl., Pulk., Kas., Taschk., Madras. |
| 13 | 9 » 20 52 | I$_f$ | III$_n$ | +3,85$^5$ | −0,18 | −0,24 | +0,06 | Wi., We., Lick, Fl. |
| 14 | 10 » 7 28 | II$_f$ | III$_n$ | −2,05$^5$ | +0,06 | +0,09$^5$ | −0,03$^5$ | Grw., Pulk., Kas., Taschk., Madras. |
| 15 | 10 » 12 4 | II$_f$ | I$_n$ | +1,24 | −0,09 | −0,08 | −0,01 | Grw., Pulk., Kas., Harv., La Pl., Rio. |
| 16 | 11 » / 11 » 15 42 smallest distance | II | IV | II +7,87 / IV +7,96 | −0,36 | −0,48 | +0,12 | Lick, Fl., Tac., AA., Harv., La Pl., Rio. |
| 17 | 11 » 19 43 | I$_{e.e.}$ | II$_n$ | +5,88 | −0,31$^5$ | −0,26 | −0,05$^5$ | Wi., We., Lick, Fl., Tac., AA. |
| 18 | 11 » 20 24 | I$_{e.e.}$ | IV$_n$ | +5,99 | −0,32 | −0,38 | +0,06 | Wi., We., Lick, Fl, |
| 19 | 12 » 23 33 | III$_f$ | II$_{w.e.}$ | −9,41 | +0,54 | +0,45 | +0,09 | Perth, HK., Tokio, Wi., We. |
| 20 | 13 » 3 57 | III$_f$ | IV$_n$ | −7,35 | +0.42 | +0,27 | +0,15 | Kasan, Taschk., Madr., HK. |
| 21 | 13 » 23 22 | III$_f$ | I$_n$ | +2,64 | −0,16 | −0,19 | +0,03 | HK., Perth, Tokio, Wi., We. |
| 22 | 14 » 1 11 | II$_f$ | I$_n$ | +1,11 | −0,09 | −0,06 | −0,03 | HK., Perth, Tokio, Wi. |
| 23 | 15 » 8 57 | I$_{e.e.}$ | II$_n$ | +5,93 | −0,32 | −0,26 | −0,06 | Grw., Pulk., Kas., Taschk., Rio. |
| 24 | 16 » 23 44 | I$_f$ | III$_n$ | +4,45$^5$ | −0,23 | −0.26 | +0,03 | HK., Pe., To., Wi., We. |
| 25 | 17 » 10 41 | II$_f$ | III$_n$ | −1,61 | +0,03 | +0,08 | −0,05 | Grw., Pulk., Kasan, La Pl., Rio. |
| 26 | 17 » 14 17 | II$_f$ | I$_n$ | +0,98 | −0,09 | −0,06 | −0,03 | Grw., Fl., Tac., AA., Harv., La Pl., Rio. |
| 27 | 18 » 22 11 | I$_{e.e.}$ | II$_n$ | +5,97 | −0,32 | −0,26 | −0,06 | Perth, To., Wi., We. |

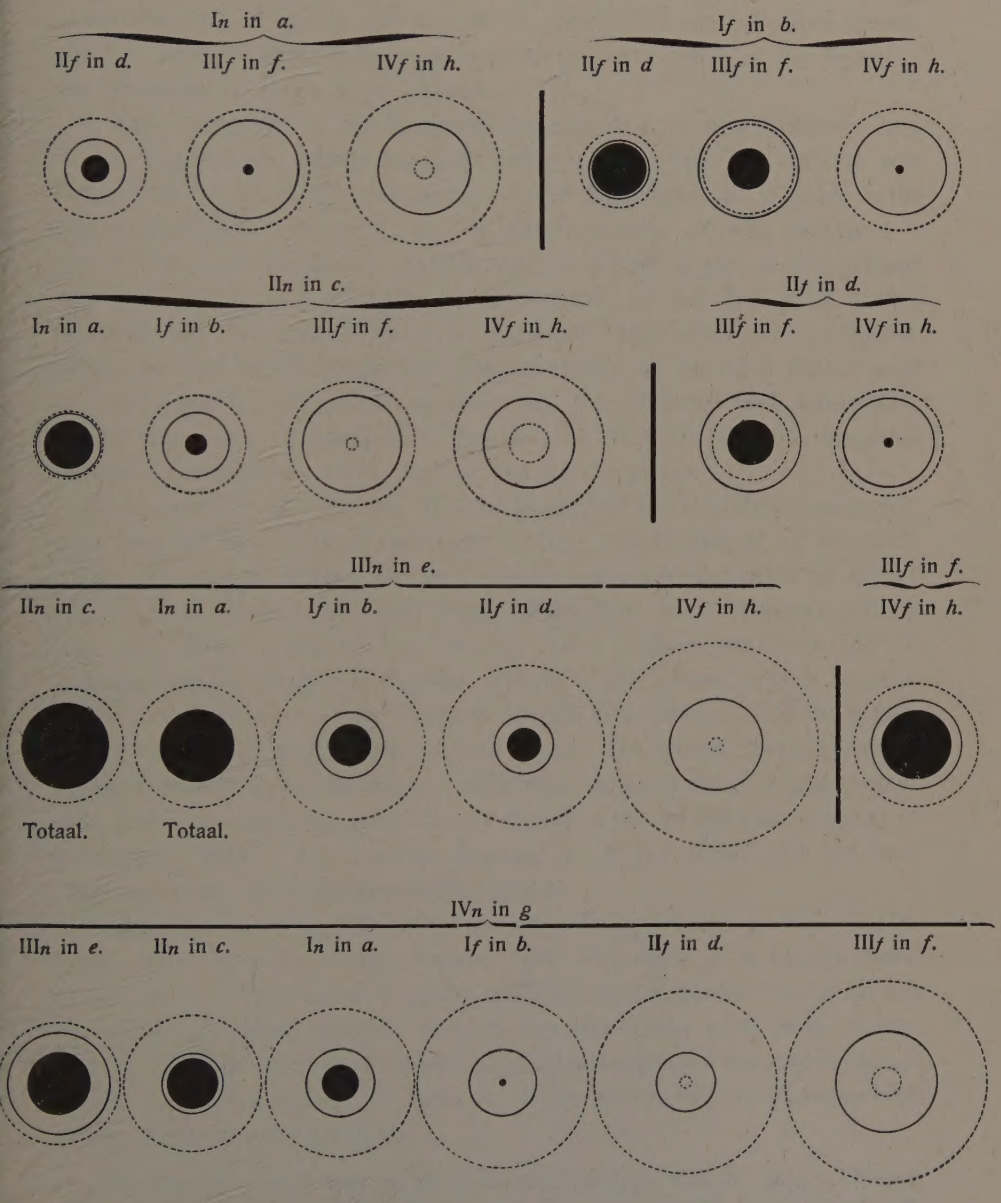
| No. | Mean time at Greenwich | $n$ = near, $f$ = far: Eclipsed satellite | Eclipsing satellite | $x=x'$ | Eclipsed satellite $y$ | Eclipsing satellite $y$ | $y-y'$ | Visible at |
|---|---|---|---|---|---|---|---|---|
| 28 | 19 April 5h15m | IV$_f$ | III$_{w.e.}$ | − 15r28 | + 0r80 | + 0r85 | − 0r05 | Kasan, Taschk., Madras. |
| 29 | 19 » 22 57 | IV$_f$ | II$_n$ | − 8,69 | + 0,48 | + 0,43 | + 0,05 | HK., Perth, Tokio, Wi., We. |
| 30 | 20 » 3 14 | III$_f$ | II$_n$ | − 9,54 | + 0,55 | + 0,45 | + 0,10 | Taschk., Madras, HK., Perth, To. |
| 31 | 21 » 1 45 | III$_f$ | I$_n$ | + 2,27 | − 0,12 | − 0,13 | + 0,01 | Madras, HK., P., Tokio. |
| 32 | 21 » 1 46 | IV$_f$ | I$_n$ | + 2,26 | − 0,05 | − 0,12 | + 0,07 | Madras, HK., P., Tokio. |
| 33 | 21 » 1 52 | IV$_f$ | III$_f$ | + 2,33 | − 0,05 | − 0,12 | + 0,07 | Madras, HK., P., Tokio. |
| 34 | 21 » 3 23 | II$_f$ | I$_n$ | + 0,845 | −0,088 | − 0,044 | − 0,044 | Taschk., Madr., HK., P., Tokio. |
| | 12 19 | IV$_f$ | II$_f$ | + 6,62 | − 0,39 | - 0,44 | + 0,05 | Grw., Pulk., Harv., La Pl., Rio. |
| 35 | 22 » 15 13 | id. gr. | dist. | 8,01 and 7.84 | − 0.32 | − 0,40 | + 0,08 | Lick, Fl., Tac., A.A., Harv., La Pl. |
| | 17 56 | » | » | + 8,88 | − 0,38 | − 0,44 | + 0,06 | We., Lick., Fl., Tac., A.A., Harv. |
| 36 | 22 » 9 41 | IV$_f$ | III$_{(o.e.)}$ | + 14,84 | − 0,66 | − 0,83 | + 0,17 | Grw., Pulk, Kasan, La Pl., Rio. |
| 37 | 22 » 11 27 | I$_{(e.e.)}$ | II$_n$ | + 6,01 | − 0,32 | − 0,26 | − 0,06 | Grw., Pulk., (Kasan), La Pl., Rio. |
| 38 | 24 » 2 41 | I$_f$ | III$_n$ | + 5,02 | − 0,255 | − 0,28 | + 0,025 | Taschk., Madr., HK., P., To. |
| 39 | 26 » 0 41 | I$_{e.e.}$ | II$_n$ | + 6,03 | − 0,33 | − 0,25 | − 0,08 | HK., P., To., Wi. |
| 40 | 27 » 7 5 | III$_f$ | II$_f$ | − 9,62 | + 0,54 | + 0,45 | + 0,09 | Bresl., Pulk., Kasan, Taschk. |
| 41 | 28 » 4 28$^s$ | I$_f$ | III$_n$ | + 1,57 | − 0,08 | − 0,09 | + 0,01 | Kasan, Taschk., Madr., HK. |
| 42 | 28 » 5 36 | II$_f$ | I$_n$ | + 0,58 | − 0,03 | − 0,065 | + 0,035 | Kasan, Taschk., Madras. |
| 43 | 28 » 13 29 | IV$_f$ | III$_n$ | + 6,41 | − 0,36 | − 0,36 | 0,00 | Grw., Tac., AA., Harv., La Pl., Rio. |
| 43* | 28 » 16 18 | II | III | II + 7,31<br>III + 7,80 | − 0,38 | − 0,49 | + 0,06 | |
| $x-x'$ diminishes gradually in absolute value, reaches its minimum 0,49 at the time assigned and then increases again. So there is no eclipse. | | | | | | | | |
| 44 | 29 » 3 45 | I$_f$ | IV$_n$ | + 0,27 | − 0,02 | − 0,08 | + 0,08 | Taschk., Madr., HK. |
| 45 | 29 » 13 57 | I$_{e.e.}$ | II$_n$ | + 6,05 | − 0,33 | − 0,26 | − 0,07 | Tac., AA., Harv., La Pl., Rio. |
| 46 | 1 May 17 5 | II$_f$ | III$_n$ | − 0,74 | − 0,01 | − 0,62 | + 0,61 | Lick, Fl., Tac., AA., Harv., La Pl. |
| 47 | 1 » 18 43 | II$_f$ | I$_n$ | + 0,44 | − 0,07 | − 0,02 | − 0,05 | Lick, Fl., Tac., AA. |
| 48 | 3 » 3 16 | I$_{e.e.}$ | II$_n$ | + 6,06 | − 0,32$^s$ | − 0,24 | − 0,08$^s$ | Taschk., Madr., HK., P., Tokio. |
| 49 | 4 » 11 7 | III$_f$ | II$_{w.e.}$ | − 9,61 | + 0,54 | + 0,45 | + 0,09 | Grw., Pulkowa. |
| 50 | 5 » 7 48 | II$_f$ | I$_n$ | + 0,31 | − 0,07 | − 0,02 | − 0,05 | Grw., Pulk., Kasan, Taschk. |
| 51 | 5 » 8 17 | III$_f$ | I$_n$ | − 0,10 | − 0,09 | − 0,01 | − 0,08 | Grw., Pulk., Kasan, (Taschk). |
| | 5 » 14 2 | III$_f$ | II$_f$ | + 4,63 | − 0,26 | − 0,26 | 0,00 | Tac., A.A., Harv., La Pl., Rio. |
| 52 | 5 » 17 57 | greatest | dist. | +6,″64 | +6,″85 | − 0 38 | − 0,36 | + 0,02 | We., Lick, Fl., Tac., AA. |
| | 5 » 21 39 | III$_f$ | II$_f$ | + 8,43 | − 0,42 | − 0,49 | + 0,07 | To., Wi., We., Lick. |

| No. | Mean time at Greenwich | $n$ = near, $f$ = far: Eclipsed satellite | Eclipsing satellite | $x = x'$ | Eclipsed satellite $y$ | Eclipsing satellite $y'$ | $y - y'$ | Visible at |
|---|---|---|---|---|---|---|---|---|
| 53 | 6 » 16h38m | $I_{e.e.}$ | $II_n$ | + 6r03 | — 0r215 | — 0r24 | + 0r025 | Lick., Fl., Tac., AA., Harvard. |
| 54 | 7 » 6 27 | $IV_n$ | $I_n$ | — 3,?7 | + 0,20 | + 0,18 | + 0,02 | Bresl. Pulk., Kasan, Taschk., Madr. |
| 55 | 7 Mei 6 37 5 | $IV_n$ | $II_n$ | — 3,18 | + 0,19 | + 0,20 | — 0,01 | Bresl. Pulk., Kasan, Taschk, Madr. |
| 56 | 8 » 7 4 | $IV_f$ | $III_n$ | + 6,90 | — 0,38 | — 0,29 | — 0,09 | Bresl., Pulk., Kasan, Taschk., Madr. |
| 57 | 8 » 20 54 | $II_f$ | $I_n$ | + 0,18 | — 0,06 | — 0,00 | — 0,06 | Wi., Wellington. |
| 58 | 8 » 23 25 | $I_n$ | $HI_n$ | — 2,04 | + 0,12 | + 0,13 | — 0,01 | Perth, Tokio, Windsor. |
| 59 | 9 » 5 29 | $I_{w.e.}$ | $III_n$ | — 5,36 | + 0,32 | + 0,34 | — 0,02 | Kasan, Taschk., Madras. |
| 60 | 10 » 6 0 | $I_n$ ($e.e.$) | $II_n$ | + 5,99 | — 0 32 | — 0,38 | + 0,06 | Kasan, Taschk., Madras. |
| 61 | 10 » 16 41 | $I_n$ | $II_n$ | — 0,97 | + 0,05 | + 0,10 | — 0,05 | Lick., Fl., Tac., AA., Harvard. |
| 62 | 11 » 0 13 | $I_n$ | $II_n$ | — 5,80 | + 0,32 | + 0,32 | 0,00 | HK., P., Tokio. |
| 63 | 11 » 15 26 | $III_f$ | $II_{w.e.}$ | — 9,46 | + 0,25 | + 0,43 | — 0,18 | Lick., Fl., Tac., AA., Harvard. |
| 64 | 12 » 9 54 | $III_f$ | $I_n$ | + 0,14 | — 0,03 | + 0,01 | — 0,04 | Grw., Pulk., Kasan, La Pl., Rio. |
| 65 | 12 » 10 0 | $II_f$ | $I_n$ | + 0,04 | — 0,38 | + 0,12 | — 0,50 | Grw., Pulk., Kasan, La Pl., Rio. |
| 66 | 12 » 10 41 | $III_f$ | $II_f$ | + 0,58 | — 0,10 | — 0,13 | + 0,03 | Grw., Pulk., La Pl., Rio. |
| 67 | 13 » 3 38 | $III_f$ | $II_f$ | + 9,32 | — 0,55 | — 0,45 | — 0,10 | Taschk., Madr., HK. |
| 68 | 13 » 19 31 | $I_n$ | $II_n$ | + 5,89 | — 0,33 | — 0,25 | — 0,08 | We., Lick., Fl. |
| 69 | 14 » 4 29 | $I_n$ | $II_n$ | + 0,10 | — 0,05 | + 0,02 | — 0,07 | Taschk., Madras. |
| 70 | 14 » 8 57 | $III_n$ ($e.e.$) | $IV_n$ | + 15 17 | — 0,87 | — 0,75 | — 0 12 | Grw., Pulk. Rio. |
| 71 | 14 » 13 46 | $I_{(w.e.)}$ | $II_n$ | — 5,93 | + 0,32 | + 0,33 | — 0 01 | Tac., AA., Harv., La Pl., Rio. |
| 72 | 15 » 9 48 | $I_f$ | $IV_n$ | + 5,59 | — 0,32 | — 0,29 | — 0,03 | Grw., Pulk., Kasan, La Pl., Rio. |
| 73 | 15 » 22 59 | $II_f$ | $IV_f$ | — 0 18 | — 0,06 | — 0,02 | — 0,04 | Perth, To., Wi. |
| 74 | 15 » 23 7 | $II_f$ | $I_n$ | — 0,09 | — 0,06 | + 0,02 | — 0,08 | Perth, To, Wi. |
| 75 | 16 » 11 59 | $I_f$ | $IV_n$ | — 5,69 | + 0,30 | + 0,24 | — 0,54 | Grw., La Pl., Rio. |
| 76 | 17 » 9 4 | $I_n$ | $II_n$ | + 5,75 | — 0,31 | — 0,23 | — 0,08 | Grw., Pulk., Kasan, La Pl., Rio. |
| 77 | 17 » 11 22 | $III_n$ ($w.e.$) | $IV_n$ | — 14,82 | + 0,80 | + 0,70 | + 0,10 | Grw., Pulk., La Pl., Rio. |
| 78 | 17 » 16 9 | $I_n$ | $II_n$ | + 1,22 | — 0,08[8] | — 0,02 | — 0,06[5] | Lick, Fl, Tac., AA., Harvard. |
| 79 | 18 » 3 12 | $I_{(w.e.)}$ | $II_n$ | — 6,00 | + 0,32 | + 0,32 | 0,00 | Taschk., Madras, HK. |
| 80 | 18 » 20 36 | $III_f$ | $II_f$ | — 8,935 | + 0,51 | + 0,41 | + 0,10 | Wi., Wellington. |
| 81 | 20 » 8 8 | $III_f$ | $II_f$ | + 9,57 | — 0,56 | — 0,45 | — 0,11 | Grw., Pulk., Kasan. |

J. A. C. OUDEMANS. "Mutual occultations and eclipses of the satellites of Jupiter in 1908." Second part: eclipses.

N. B. The continuous circles show the contour of the satellites, the dotted circles represent the penumbra.

Scale 1 : 314 250 000. 12mm = 1" heliocentric.



**Physics.** — "*Contribution to the knowledge of the ψ-surface of* VAN DER WAALS. XI. *A gas that sinks in a liquid.*" By Prof. H. KAMERLINGH ONNES. Communication N°. 96 from the Physical Laboratory of Leiden.

If we have an ideal gas and an incompressible liquid without vapour tension, in which the gas does not dissolve, the gas will gather above the liquid under the action of gravity, if the pressure is sufficiently low, whereas the compressed gas will sink in the liquid if the pressure is made high enough.

I have observed a phenomenon approaching to this fictitious case in an experiment which roughly came to this, that helium gas was compressed more and more above liquid hydrogen till it sank in the liquid hydrogen. Roughly, for so simple a case as was premised is not to be realized. Every experiment in which a gas is compressed above a liquid, is practically an application of the theory of binary mixtures of VAN DER WAALS. In such an experiment the compressibility of the liquid phase and the solubility of gas and liquid inter se may not be neglected, as generally the pressure will even have to be increased considerably before the density of the gas-phase becomes comparable with that of the liquid phase.

If the theory of VAN DER WAALS is applied to suchlike experiments, the question lies at hand whether in the neighbourhood of the plait-point phenomena where gas and liquid approach each other so closely that of the ordinary gas and liquid state they have retained nothing but the name, perhaps on account of a higher proportion of the substance with greater molecular weight [1]) the phase, which must be called the gas phase, may become specifically heavier than the phase, which must be called the liquid phase. On closer investigation it appears however, to be due to relations between the physical properties and the chemical constitution (so also the molecular weight) of substances, that a liquid phase floating on a gas phase has not been observed even in this favourable region.

I was the more struck with an irregularity which I came across when experimenting with helium and hydrogen in a closed metal vessel, as I thought that I could explain it by the above mentioned not yet observed phenomenon, and so the conviction took hold of me, that at — 253° and at a pressure of 60 atmospheres the gaslike phase which chiefly consists of helium, sinks in the liquid phase which chiefly consists of hydrogen.

[1]) The limiting case is that in the ψ-surface construed with the unity of weight the projection of the nodal chord on the $xv$-plane runs parallel to the line $v = 0$.

In order to ascertain myself of this I compressed by means of the mercury compressor described in Communication N°. 54 a mixture of about one part of helium and 6 parts of hydrogen in a glass tube, which had a capillary inflow tube at the top, and a capillary outlet tube at the bottom, and which was merged in liquid hydrogen.

Up to 49 atmospheres the liquid hydrogen was seen to deposit from the gas mixture, bounded by a distinct hollow meniscus against the helium. At 49 atmospheres the helium, or properly speaking the gas phase consisting chiefly of helium, went down just as water through oil, and remained on the bottom as a large drop. With further compression to 60 atmospheres and decrease of pressure to 32 atmospheres the volume of the bubble appeared to follow the change of the pressure as that of a gas. At 32 atmospheres the bubble rose again. By changing the pressure the bubble was made to rise and descend at pleasure.

The closer investigation of these phenomena in connection with the isotherms of helium and the $\psi$-surfaces of $H_2$ and He is an extensive work, so that in anticipation of the results which most likely will be definitely drawn up only much later, I feel justified in confining myself to this sketchy communication.

One remark may be added now. It appears that the $b$ of helium must be small, from which follows again that $a$ must have an exceedingly small value, because the critical temperature, if it exists, must lie very low. In this direction points also a single determination of the plaitpoint of a mixture of helium and hydrogen which I have already made. Whether $a$ has really a positive value, whether it is zero, or whether (what is also conceivable) $a$ is negative, will have to be decided by the determination of the isotherms of helium.

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**(December 21, 1906).**